Exercise 9

A Rate change relating to Cartesian Equations

1
Solution

Given $16x^2 + 9y^2 = 144$ .... (1)
Substituting $x = \sqrt{5}$ into (1).

$$y = \pm\sqrt{\frac{144 - 16(5)}{9}}$$
$$= \pm\frac{8}{3}$$

Since $\frac{\mathrm{d}y}{\mathrm{d}t}$, $x > 0$. Hence for $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$, y should be negative.

$\therefore\ y = -\frac{8}{3}$

Differentiate (1) with respect to x:

$$32x + 18y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-32x}{18y}$$
$$= -\frac{16x}{9y}$$

At $\left(\sqrt{5}, -\frac{8}{3}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-16\sqrt{5}}{3(-8)}$
$$= \frac{2\sqrt{5}}{3}$$

Using chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

Given that $\frac{\mathrm{d}y}{\mathrm{d}t} = 2\text{ cm s}^{-1}$,

$$2 = \frac{2\sqrt{5}}{3} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{\sqrt{5}}$$

At $\left(\sqrt{5}, -\frac{8}{3}\right)$, its rate of increase is $\frac{3}{\sqrt{5}}$ cm/s.

Alternative Method

Differentiate (1) with respect to t :

$$32x\frac{\mathrm{d}x}{\mathrm{d}t}+18y\frac{\mathrm{d}y}{\mathrm{d}t}=0 \text{ .... (2)}$$

Substituting $x=\sqrt{5}$, $y=-\frac{8}{3}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=2$ into (2).

$$32\sqrt{5}\frac{\mathrm{d}x}{\mathrm{d}t}+18\left(-\frac{8}{3}\right)(2)=0$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{3}{\sqrt{5}}$$

At $\left(\sqrt{5},-\frac{8}{3}\right)$, its rate of increase is $\frac{3}{\sqrt{5}}$ cm/s.

2
Solution

Given $\frac{1}{u}+\frac{1}{v}=\frac{1}{20}$ .... (1)

Differentiate (1) with respect to t

$$-\frac{1}{u^2}\frac{du}{dt}-\frac{1}{v^2}\frac{dv}{dt}=0 \text{ .... (2)}$$

Substituting $u=60$ into (2)

$$\frac{1}{60}+\frac{1}{v}=\frac{1}{20}$$
$$\frac{1}{v}=\frac{1}{20}-\frac{1}{60}$$
$$\frac{1}{v}=\frac{1}{30}$$
$$v=30$$

Substituting $v=30$ and $\frac{du}{dt}=-2$ into (2).

$$\frac{1}{(60)^2}(-2)-\frac{1}{(30)^2}\frac{dv}{dt}=0$$
$$\frac{1}{1800}-\frac{1}{900}\frac{dv}{dt}=0$$
$$\frac{dv}{dt}=\frac{900}{1800}$$
$$=\frac{1}{2}$$

$\therefore$ the rate of increase of v when $u=60$ units v is $\frac{1}{2}$ units/s.

3
Solution

Let $A(x)=\sqrt{1+\frac{1}{x^2}}$ be $A=\sqrt{1+\frac{1}{x^2}}$

Differentiate A with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{1}{2}\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}\left(-\frac{2}{x^3}\right)$$

$$=-\frac{1}{x^3\sqrt{1+\frac{1}{x^2}}}\text{............................ (1)}$$

Given that when $x=k,$ the rate of decrease of A is k times the rate of increase of x,

i.e. $\frac{\mathrm{d}A}{\mathrm{d}t}=-k\frac{\mathrm{d}x}{\mathrm{d}t}$

$\therefore$ $$\frac{\mathrm{d}A}{\mathrm{d}t}=-k\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}A}{\mathrm{d}x}=-k\text{............................ (2)}$$

Substitute (1) into (2)

$$-\frac{1}{x^3\sqrt{1+\frac{1}{x^2}}}=-k$$

When $x=k$

$$-\frac{1}{k^3\sqrt{1+\frac{1}{k^2}}}=-k$$

$$\frac{1}{k\sqrt{k^2+1}}=k$$

$$k^4\sqrt{k^2+1}=1$$

From GC, since $k>0,$ $k=0.905$ (3 s.f.)

Exercise 9

A Rate of change relating to Cartesian Equations

4
Solution

(a) Given $x = p(\cos 2\theta - 2\cos\theta)$ .... (1)

and $y = p\sin\frac{3\theta}{2}$ .... (2)

Differentiate (1) with respect to θ

$$\frac{dx}{d\theta} = p(-2\sin 2\theta + 2\sin\theta)$$

Differentiate (2) with respect to θ

$$\frac{dy}{d\theta} = \frac{3}{2}p\cos\frac{3\theta}{2}$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\frac{3}{2}p\cos\frac{3\theta}{2}}{p(-2\sin 2\theta + 2\sin\theta)}$$

$$= -\frac{3\cos\frac{3\theta}{2}}{4(\sin 2\theta - \sin\theta)}$$

When $x = -p$, substitute $x = -p$ into (1).

$-p = p(\cos 2\theta - 2\cos\theta)$

$\cos 2\theta - 2\cos\theta + 1 = 0$

$2\cos^2\theta - 2\cos\theta = 0$

$\cos\theta(\cos\theta - 1) = 0$

$\cos\theta = 0$ or $\cos\theta = 1$

$\therefore\ \theta = \frac{\pi}{2}$ or $\theta = 0$ or 2π [rejected since $0 < \theta < \pi$]

When $\theta = \frac{\pi}{2}$,

$$\frac{dy}{dx} = -\frac{3\cos\frac{3}{2}\left(\frac{\pi}{2}\right)}{4\left[\sin 2\left(\frac{\pi}{2}\right) - \sin\frac{\pi}{2}\right]}$$

$$= -\frac{3\sqrt{2}}{8}$$

$$\frac{dx}{d\theta} = p(-2\sin 2\theta + 2\sin\theta)$$

$$= 2p$$

Using the Chain Rule,

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$

$$= 2p \times \frac{\pi}{3}$$

$$= \frac{2p\pi}{3}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= -\frac{3\sqrt{2}}{8} \times \frac{2p\pi}{3}$$

$$= -\frac{\sqrt{2}}{4} p\pi$$

$\therefore$ y changes at a rate of $-\frac{\sqrt{2}}{4} p\pi$ unit per second.

(b) Using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$-\frac{3\cos\frac{3\theta}{2}}{4(\sin 2\theta - \sin\theta)} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Given $\frac{dy}{dt} = -1.5$ and when $\theta = 1$,

$$-\frac{3\cos\frac{3}{2}}{4(\sin 2 - \sin 1)} = -1.5 \times \frac{dt}{dx}$$

$$\frac{dx}{dt} = 4\sin\frac{1}{2}$$

Now differentiate $(x + y)$ with respect to t

$$\frac{d}{dt}(x + y) = \frac{dx}{dt} + \frac{dy}{dt}$$

$$= 4\sin\frac{1}{2} - 1.5$$

$$= 0.4177021544$$

$\therefore x + y$ changes at a rate of 0.4177 unit per second.

5
Solution

(a) Given $x = a\sin^3\theta$ .... (1)

and $y = a\cos^3\theta$ .... (2)

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}x}\\ &= \frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta}\\ &= -\cot\theta\end{aligned}$$

(b) Equation of tangent at the point P:

$$y - a\cos^3\theta = -\cot\theta(x - a\sin^3\theta)$$

$$y - a\cos^3\theta = -\frac{\cos\theta}{\sin\theta}(x - a\sin^3\theta)$$

$$y\sin\theta + x\cos\theta = a\sin^3\theta\cos\theta + a\cos^3\theta\sin\theta$$

$$y\sin\theta + x\cos\theta = a\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$$

$y\sin\theta + x\cos\theta = a\sin\theta\cos\theta$ .... (1)

$\therefore$ the equation of tangent to the curve at the point P is $y\sin\theta + x\cos\theta = a\sin\theta\cos\theta$. (Shown)

(c) When the tangent at the point P meets the x-axis, i.e. $y = 0$.

Substitute $y = 0$ into (1).

$$\begin{aligned}(0)\sin\theta + x\cos\theta &= a\sin\theta\cos\theta\\ x &= a\sin\theta\end{aligned}$$

$\therefore$ $L = (a\sin\theta, 0)$

When the tangent at the point P meets the y-axis, i.e. $x = 0$.

Substitute $x = 0$ into (1).

$$\begin{aligned}y\sin\theta + (0)\cos\theta &= a\sin\theta\cos\theta\\ y &= a\sin\theta\end{aligned}$$

$\therefore M = (0, a\cos\theta)$

$$\begin{aligned}LM &= \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\\ &= \sqrt{a^2(\sin^2\theta + \cos^2\theta)}\\ &= a\end{aligned}$$

$\therefore$ LM is dependent of a. (Shown)

(c) Let $z = \dfrac{x}{y}$

$$= \frac{a\sin^3\theta}{a\cos^3\theta}$$

$$= \tan^3\theta$$

$$\frac{dz}{d\theta} = 3\tan^2\theta\sec^2\theta$$

Using the Chain Rule,

$$\frac{dz}{dt} = \frac{dz}{d\theta} \times \frac{d\theta}{dt}$$

When $\theta = \dfrac{\pi}{6}$ and given that rate of increase in θ is 0.05 radian s^{-1}, i.e. $\dfrac{d\theta}{dt} = 0.05$

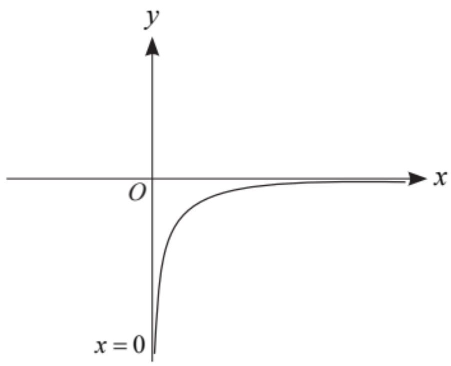
$$= \left(3\tan^2\frac{\pi}{6}\sec^2\frac{\pi}{6}\right) \times 0.05$$

$$= 0.0667$$

The rate at which $\dfrac{x}{y}$ is changing when $\theta = \dfrac{\pi}{6}$ is 0.0667 radians s^{-1}

6
Solution

(a)



Learning Point :

As $\theta \to 0,\ x = -\ln(\cos\theta) \to 0,\ y = \ln(\sin\theta) \to -\infty$

As $\theta \to \frac{\pi}{2},\ x = -\ln(\cos\theta) \to +\infty,\ y = \ln(\sin\theta) \to 0$

(b) Given $x = \ln\left(\frac{1}{\cos\theta}\right)$

$= -\ln(\cos\theta)$ .... (1)

and $y = \ln(\sin\theta)$ .... (2)

Differentiate (1) with respect to θ

$$\frac{dx}{d\theta} = \frac{\sin\theta}{\cos\theta}$$

Differentiate (2) with respect to θ

$$\frac{dy}{d\theta} = \frac{\cos\theta}{\sin\theta}$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \cot^2\theta$$

When $\theta = \frac{\pi}{4}$, substitute $\theta = \frac{\pi}{4}$ into (1) and (2)

From (1): $x = -\ln\left(\cos\frac{\pi}{4}\right)$

$= \frac{1}{2}\ln 2$

From (2): $y = \ln\left(\sin\frac{\pi}{4}\right)$

$= -\frac{1}{2}\ln(2)$

From (3): $\frac{\mathrm{d}y}{\mathrm{d}x} = \cot^2\frac{\pi}{4}$

$= 1$

$\therefore$ the coordinates are $\left(\frac{1}{2}\ln 2,\ 1\right)$ and the gradient at $\theta = \frac{\pi}{4}$ is 1.

Equation of tangent at the point where $\theta = \frac{\pi}{4}$ is

$$y - \left(-\frac{1}{2}\ln 2\right) = x - \frac{1}{2}\ln 2$$

$$y + \frac{1}{2}\ln 2 = x - \frac{1}{2}\ln 2$$

$$y = x - \ln 2$$

(c) Let A be the area of triangle OBP

$$A = \frac{1}{2}\times 1 \times y$$

$$= \frac{1}{2}\times 1\times(-\ln(\sin\theta))$$

$$= -\frac{1}{2}\ln(\sin\theta)$$

Differentiate A with respect to θ.

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = -\frac{1}{2}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= -\frac{1}{2}\cot\theta$$

Using the Chain Rule,

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$

$$= -\frac{1}{2}\cot\theta \times 2$$

$$= -\cot\theta$$

When $\theta = \frac{\pi}{3}$

$$\frac{dA}{dt} = -\left(\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}\right)$$

$$= -\frac{1}{2} \div \frac{\sqrt{3}}{2}$$

$$= -\frac{1}{\sqrt{3}} \text{ units}^2/\text{s}$$

Rate of change of area A

$= \frac{dA}{dt}$

$= \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= \left[-\frac{1}{2}\left(\frac{\cos\theta}{\sin\theta}\right)\right] \times 2 \qquad \triangleleft \text{ given } \frac{d\theta}{dt} = 2$

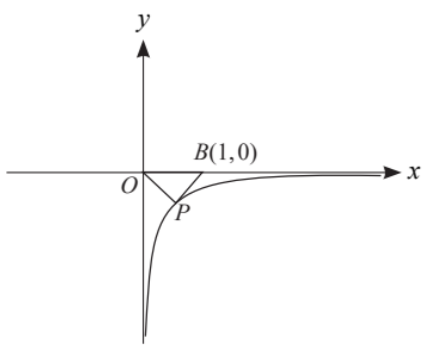
$= -\cot\theta$

When $\theta = \frac{\pi}{3}$

$$\frac{dA}{dt} = -\left(\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}\right)$$

$$= -\frac{1}{2} \div \frac{\sqrt{3}}{2}$$

$$= -\frac{1}{\sqrt{3}} \text{ units}^2/\text{s}$$



The rate of the area of triangle OBP is decreasing when $\theta = \frac{\pi}{3}$ is $\frac{1}{\sqrt{3}}$ units^2/s.

Exercise 9

C Applications

7
Solution

Let the height and the radius of the cylinder, V be h and x respectively.

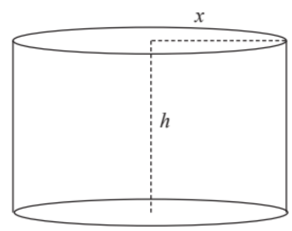
$V = \pi x^2 h$ .... (1)

Given that the height of a cylinder is twice the radius, i. e. $h = 2x$ .... (2)

Substitute (2) into (1)

$V = \pi x^2 (2x)$

$= 2\pi x^3$.......................... (3)



Differentiating V respect to x

$$\frac{dV}{dx} = 6\pi x^2$$

When $x = 2$, substitute $x = 2$ into (3)

$$\frac{dV}{dx} = 6\pi(2)^2$$
$$= 24\pi$$

Given x is increasing at a rate of 0.1 cm/s, i.e. $\frac{dx}{dt} = 0.1$

Using the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$
$$= 24\pi \times 0.1$$
$$= 2.4\pi$$

$\therefore$ the rate of change of volume is 2.4 π cm^3/s.

8
Solution

Let the circumference of the blot be C.

$$C = \pi D \quad \lhd \text{ formula for circumference of a circle} = \pi \times (\text{diameter of circle})$$

Differentiating both sides wrt t

$$\frac{dC}{dt} = \pi \frac{dD}{dt}$$

Given that $\frac{dD}{dt} = 0.25$ cm/s,

$$\therefore \quad \frac{dC}{dt} = 0.25\pi$$
$$= 0.7854 \text{ (4 d.p.)}$$

$\therefore$ the rate of change of the circumference of the blot is 0.7854 cm/s.

$$D = 2r, \quad \text{where } r \text{ is the radius}$$

$$\frac{dD}{dt} = 2\frac{dr}{dt}$$

Given that $\frac{dD}{dt} = 0.25$

$$\frac{dr}{dt} = \frac{1}{2}(0.25)$$
$$= 0.125$$

Let the area of the blot be C.

$$A = \pi r^2$$

Differentiating both sides wrt t,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

When $r = 1.5$ cm,

$$\frac{dA}{dt} = 2\pi(1.5)(0.125)$$
$$= 1.1781 \text{ cm}^2/\text{s} \quad \text{(4 d.p.)}$$

$\therefore$ the rate of the area of the blot is $1.1781 \text{ cm}^2/\text{s}$.

9
Solution

(a) Let the radius and the volume of a spherical balloon be r and V respectively.

$$V = \frac{4}{3}\pi r^3 \text{ .... (1)}$$

Differentiate V with respect to r

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$$

Air is being blown into a spherical balloon at a constant rate of 12 cm^3 per minute, $\frac{\mathrm{d}V}{\mathrm{d}t} = 12$.

Using the Chain Rule,

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\mathrm{d}r}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} \\
&= \frac{1}{4\pi r^2} \times 12 \\
&= \frac{12}{4\pi r^2} \\
&= \frac{3}{\pi r^2}
\end{aligned}$$

At $r = 5$

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{3}{\pi (5)^2} \\
&= \frac{3}{25\pi} \\
&= 0.0382
\end{aligned}$$

$\therefore$ the rate at which the radius of the balloon is changing when the radius is 5 cm is 0.0382 cm/min.

(b) Let the radius and the area of a spherical balloon be r and A respectively.

$$A = 4\pi r^2$$

Differentiate V with respect to r

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} \qquad \triangleleft \text{ substitute } \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{\pi r^2} \text{ in } \textbf{(a)}$$

$$\begin{aligned}
&= (8\pi r) \times \frac{3}{\pi r^2} \\
&= \frac{24}{r} \text{ .... (2)}
\end{aligned}$$

Given that air is blown into spherical balloon at a constant rate of 12 cm^3 per minute

In 10 min, the volume of the balloon $V = 12 \times 10 = 120 \text{ cm}^3$.

Substitute $V = 120$ into (1).

$$\frac{4}{3}\pi r^3 = 120$$

$$r = \sqrt[3]{\frac{90}{\pi}}$$

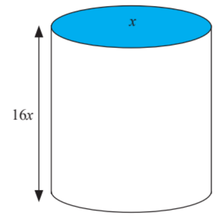
Substitute $r = \sqrt[3]{\frac{90}{\pi}}$ into (2).

At 10 minutes, $\frac{\mathrm{d}A}{\mathrm{d}t} = 24\sqrt[3]{\frac{\pi}{90}}$

$= 7.84 \text{ cm/min}$

$\therefore$ the rate at which the surface area of the balloon is changing after 10 minutes is 7.84 cm/min.

10
Solution



(a) Let h be the height of the cylinder.

$h = 16x$ .... (1)

Differentiating (1) respect to t

$\frac{\mathrm{d}h}{\mathrm{d}t} = 16\frac{\mathrm{d}x}{\mathrm{d}t}$ .... (2)

Given when $x = 4$, area of the cross-section is increasing at a rate of $0.02\text{ cm}^2/\text{s}$, i.e. $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.02$.

Substitute $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.02$ into (2)

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 16(0.02)$$
$$= 0.32\text{ cm/s}$$

$\therefore$ the rate of increase of the height of the cylinder when $x = 4$ is 0.32 cm/s.

(b) $V = (\text{Area of cross-section}) \times \text{height}$

$= (16x) \times x$

$= 16x^2$

Differentiating V respect to x

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 32x$$

Using the Chain Rule,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$= 32x \times 0.02$$
$$= 0.64x$$

When $x = 4$,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 32(4)(0.02)$$
$$= 2.56$$

$\therefore$ the rate of increase of the volume of the cylinder when $x = 4$ is $2.56\text{ cm}^3/\text{s}$

(c) $x = \pi r^2$

When $x = 5$,

$$5 = \pi r^2$$

$$\sqrt{\frac{5}{\pi}} = r$$

$$\frac{\mathrm{d}x}{\mathrm{d}r} = 2\pi r$$

Given $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.02 \text{ cm}^2/\text{s}$ when $x = 4$ and $r = \sqrt{\frac{5}{\pi}}$

Using the Chain Rule,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$0.02 = 2\pi\left(\sqrt{\frac{5}{\pi}}\right) \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0.00252$$

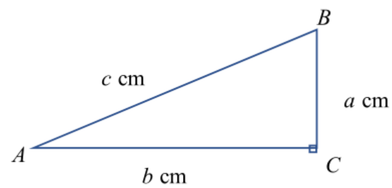
$\therefore$ the rate of increase of the radius of the cylinder when $x = 4$ is $0.00252 \text{ cm}/\text{s}$

11
Solution

(a) Let A be the area of the triangle.

Given that the triangle has a fixed area of 100 cm^2, i.e. $A = 100$

$$\frac{1}{2}ab = 100$$
$$b = \frac{200}{a}$$



(b) By Pythagoras Theorem

$$c^2 = a^2 + b^2$$
$$= a^2 + \frac{200^2}{a^2} \quad \text{.............................. (1)}$$

Differentiate both sides with respect to t

$$2c\frac{\mathrm{d}c}{\mathrm{d}t} = 2a\frac{\mathrm{d}a}{\mathrm{d}t} - \frac{2(200)^2}{a^3}\frac{\mathrm{d}a}{\mathrm{d}t} \quad \text{.............................. (2)}$$

When $a = 20$, $\quad c = \sqrt{20^2 + \frac{200^2}{20^2}}$
$$= \sqrt{500}$$

Substitute $a = 20,\ c = \sqrt{500}$ and into $\frac{\mathrm{d}a}{\mathrm{d}t} = 3$ into (2)

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{1}{2\sqrt{500}}\left(2(20)(3) - \frac{2(200)^2}{20^3}(3)\right)$$
$$= \frac{90}{20\sqrt{5}}$$
$$= \frac{9}{2\sqrt{5}}$$

$\therefore$ the c is increasing at rate of $\frac{9}{2\sqrt{5}}$ cm s^{-1}

Alternative Method

Pythagoras Theorem

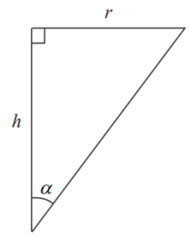
$$c^2 = a^2 + b^2$$
$$c = \sqrt{a^2 + \frac{200^2}{a^2}}$$
$$\frac{\mathrm{d}c}{\mathrm{d}a} = \frac{1}{2}\left(a^2 + \frac{200^2}{a^2}\right)^{-\frac{1}{2}}(2a + 200^2(-2a^{-3}))$$

When $a = 20$,

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{\mathrm{d}c}{\mathrm{d}a} \times \frac{\mathrm{d}a}{\mathrm{d}t}$$
$$= \frac{1}{2}\left(a^2 + \frac{200^2}{20^2}\right)^{-\frac{1}{2}}\left(40 + 200^2\left(\frac{-2}{20^3}\right)\right)(3)$$
$$= 2.01$$

12
Solution

(a) Let the depth of water at t minutes be h m.



$$\tan\alpha = \frac{r}{h}$$

$$0.5 = \frac{r}{h} \qquad \triangleleft \tan\alpha = 0.5 \text{ (given)}$$

$$r = \frac{h}{2} \text{ .... (1)}$$

$$V = \frac{1}{3}\pi r^2 h \text{ .... (2)}$$

Substitute (1) into (2)

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \frac{\pi h^3}{12}$$

Differentiate V with respect to h

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi h^2}{4}$$

Given $V = 3\text{m}^3$, substitute $V = 3$ into (2)

$$3 = \frac{1}{3}\pi r^2 h$$

$$h = \sqrt[3]{\frac{36}{\pi}}$$

Using the chain rule,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \frac{4}{\pi h^2}\times\frac{1}{10}$$

$$= \frac{2}{5\pi h^2}$$

When $h = \sqrt[3]{\frac{36}{\pi}}$,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2}{5\pi\left(\frac{36}{\pi}\right)^{\frac{2}{3}}}$$

$$= 0.0251 \text{ m min}^{-1}$$

$\therefore$ the rate of increase of the depth of water when the volume of water is 3 m^3 is 0.0251 m min^{-1}.

(b) Let A be the top surface area of the water in the container.

$A = 4\pi r^2$ .... (3)

Substitute (1) into (3)

$$A = 4\pi\left(\frac{h}{2}\right)^2$$
$$= \pi h^2$$

Differentiate A with respect to t

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi h\frac{\mathrm{d}h}{\mathrm{d}t}$$

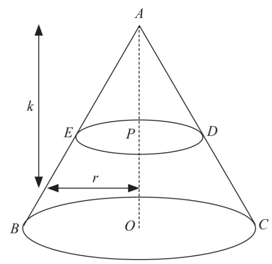
When $h = \sqrt[3]{\frac{36}{\pi}}$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = 0.0251$,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi\left(\sqrt[3]{\frac{36}{\pi}}\right)(0.0251)$$
$$= 0.355$$

$\therefore$ the rate of the top surface area of the water in the container changes at this instant is 0.355 m min^{-1}.

13
Solution

Refer to the diagram.



Let BC be the diameter of the top of the cylindrical part of container with O as the centre and let ED be the diameter of the lid with P as the centre. The lines BE and CD are extended to meet at A as shown in the diagram.

The points A, B, C, D and E lie in the same plane and ABC and AED form two right cones.

Let $OB = 3$ cm, $PE = 1.5$ cm.

By Triangle Midpoint Theorem, $BE = EA = 2.5$ cm. Hence $BA = 5$ cm

Use pythagoras theorem, $OA = \sqrt{BA^2 - OB^2}$

$\therefore\ OA = 4$ cm.

Let r be the radius of the liquid surface, and k be the vertical distance from the liquid surface to A.

Using similar triangles, $\dfrac{r}{k} = \dfrac{3}{4}$

$$r = \frac{3}{4}k$$

Volume of liquid above level $BC, V = \dfrac{1}{3}\pi(3^2)(4) - \dfrac{1}{3}\pi r^2 k$ .... (1)

Substitute $r = \dfrac{3}{4}k$ into (1)

$$V = \frac{1}{3}\pi(3^2)(4) - \frac{3}{16}\pi k^3$$

Differentiate V with respect to k

$$\frac{\mathrm{d}V}{\mathrm{d}k} = -\frac{9}{16}\pi k^2$$

When the liquid level is 1 cm from the lid of the container, $k = 3$

$$\therefore\ \frac{\mathrm{d}V}{\mathrm{d}k} = -\frac{9}{16}\pi(3)^2$$

Using chain rule,

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}k}{\mathrm{d}V}$$

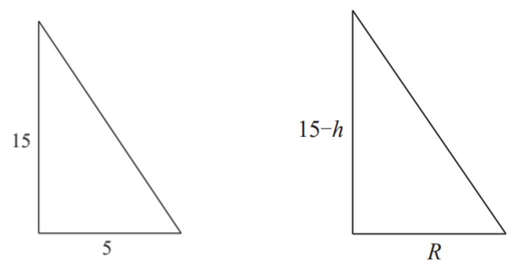
$$= 90\pi \times \left(-\frac{16}{9\pi(3)^2}\right)$$

$$= -\frac{160}{9}$$

Since k is decreasing at $\dfrac{160}{9}$ cm/s, it follows that the liquid level in the can is increasing at $\dfrac{160}{9}$ cm/s when it is 1 cm from the top of the container.

14
Solution

Let the depth of the water be h and the radius of the water level at that point be R.



Using similiar triangle,

$$\frac{15-h}{15}=\frac{R}{5}$$

$$R=\frac{15-h}{3} \text{ .... (1)}$$

Let V be the volume of water in the cone

$$V=\frac{1}{3}\pi(5)^2(15)-\frac{1}{3}\pi R^2(15-h) \text{ .... (2)}$$

Substitute (1) into (2)

$$V=\frac{1}{3}\pi(5)^2(15)-\frac{1}{3}\pi\left(\frac{15-h}{3}\right)^2(15-h)$$

$$=\frac{1}{3}\pi(5)^2-\frac{1}{27}\pi(15-h)^3$$

Differentiate V with respect to h

$$\frac{dV}{dh}=-\frac{3}{27}\pi(15-h)^2(-1)$$

$$=\frac{\pi(15-h)^2}{9}$$

Given that liquid is leaking at a rate of $10\text{ cm}^3\text{s}^{-1}$, $\frac{dV}{dt}=-\pi\text{ cm}^3\text{s}^{-1}$.

Using the Chain Rule,

$$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$$

$$-\pi=\frac{\pi(15-h)^2}{9}\times\left(-\frac{1}{16}\right)$$

$$144=(15-h)^2$$

$$\pm12=5-h$$

$$5-h=12 \quad\text{or}\quad 5-h=-12$$

$$h=-7 \text{ (Rejected, } h>0) \quad h=17$$

$\therefore$ the depth of liquid is 17 cm.

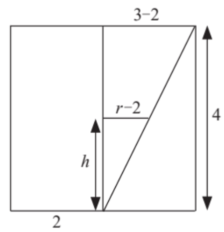
15
Solution

(a) Using similiar triangle,

$$\frac{r-2}{h}=\frac{3-2}{4}$$

$$r-2=\frac{h}{4}$$

$$r=2+\frac{h}{4} \text{ .... (1)} \quad \text{(Shown)}$$



(b) Using the volume of a frustum, $V=\frac{1}{3}\pi(r_1^2+r_2^2+r_1r_2)$

Substitute $r_1=2$ and $r_2=r$

$\therefore$ volume of water, $V=\frac{1}{3}\pi(2^2+r^2+2r)h$ .... (2)

Substitute (1) into (2)

$$V=\frac{1}{3}\pi\left(4+\left(2+\frac{h}{4}\right)^2+2\left(2+\frac{h}{4}\right)\right)h$$

$$=\frac{1}{3}\pi h\left(4+4+h+\frac{h^2}{16}+4+\frac{h}{2}\right)$$

$$=\frac{1}{3}\pi\left(\frac{h^3}{16}+\frac{3h^2}{2}+12h\right)$$

Differentiate V with respect to h

$$\frac{dV}{dh}=\frac{1}{3}\pi\left(\frac{3h^2}{16}+3h+12\right)$$

Given that water is poured at a rate of 9 m^3 per minute into an open container, i.e. $\frac{dV}{dt}=9$.

Using the Chain Rule,

$$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$$

$$9=\frac{1}{3}\pi\left(\frac{3h^2}{16}+3h+12\right)\times\frac{dh}{dt}$$

When $h=1$,

$$9=\frac{1}{3}\pi\left(\frac{3(1)^2}{16}+3(1)+12\right)\times\frac{dh}{dt}$$

$$9=\frac{81\pi}{16}\times\frac{dh}{dt}$$

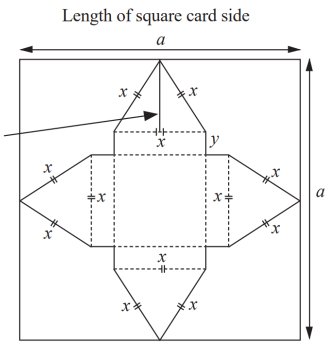
$$\frac{dh}{dt}=\frac{16}{9\pi}$$

$\therefore$ the rate of increase of the depth of water is $\frac{16}{9\pi}$ m min^{-1}.

Exercise 9

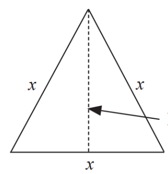
D Mixed Practice

16
Solution



(a) Use Pythagoras' Theorem

$$\text{Height of triangle} = \sqrt{x^2 - \left(\frac{1}{2}x\right)^2}$$
$$= \sqrt{\frac{3}{4}x^2}$$
$$= \frac{\sqrt{3}}{2}x$$



$$\text{Length of square card side} = x + 2y + 2(\text{height of triangle})$$

$$a = x + 2y + 2\left(\frac{\sqrt{3}}{2}x\right) \quad \triangleleft \text{ make } y \text{ as a subject}$$

$$y = \frac{1}{2}\left(a - \left(1+\sqrt{3}\right)x\right) \qquad \text{(Shown)}$$

$$\text{Height of pyramid} = \sqrt{\left(\frac{\sqrt{3}}{2}x\right)^2 - \left(\frac{1}{2}x\right)^2}$$
$$= \sqrt{\frac{1}{2}x^2}$$
$$= \frac{1}{\sqrt{2}}x$$

$$V = \text{Volume cuboid} + \text{volume of pyramid}$$
$$= x^2y + \frac{1}{3}x^2\left(\frac{1}{\sqrt{2}}x\right)$$
$$= x^2\left[\frac{1}{2}\left(a - \left(1+\sqrt{3}\right)x\right)\right] + \frac{1}{3}x^2\left(\frac{1}{\sqrt{2}}x\right)$$
$$= \frac{1}{2}ax^2 - \frac{1}{2}\left(1+\sqrt{3}\right)x^3 + \frac{1}{3\sqrt{2}}x^3$$
$$= \frac{1}{2}ax^2 + \left(\frac{1}{3\sqrt{2}} - \frac{1}{2} - \frac{\sqrt{3}}{2}\right)x^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$\therefore$ the volume of the container is $\frac{1}{2}ax^2 + \left(\frac{1}{3\sqrt{2}} - \frac{1}{2} - \frac{\sqrt{3}}{2}\right)x^3$ m

(b) Differentiate (1) with respect to x

$$\frac{\mathrm{d}V}{\mathrm{d}x} = ax + \left(\frac{1}{3\sqrt{2}} - \frac{1}{2} - \frac{\sqrt{3}}{2}\right)3x^2$$

At stationary, $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$

i.e. $$ax + \left(\frac{1}{3\sqrt{2}} - \frac{1}{2} - \frac{\sqrt{3}}{2}\right)3x^2 = 0$$

$$x\left[a + \left(\frac{1}{3\sqrt{2}} - \frac{\sqrt{3}}{2}\right)3x\right] = 0$$

$$\left(\frac{1}{3\sqrt{2}} - \frac{\sqrt{3}}{2}\right)3x = -a \quad \text{or} \quad x = 0 \ \text{(Rejected, } x > 0\text{)}$$

$$x = \frac{a}{3\left(\frac{\sqrt{3}}{2} - \frac{1}{3\sqrt{2}}\right)}$$

$$x = \frac{6\sqrt{2}a}{36\sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{3\sqrt{2}}\right)}$$

$$= \frac{2\sqrt{2}a}{3\sqrt{6} - 2}$$

$\therefore$ the maximum value of x is $\frac{2\sqrt{2}a}{3\sqrt{6} - 2}$.

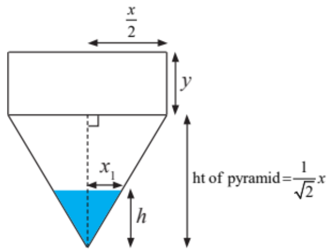
(c) Let x_1 be side of the small square base pyramid

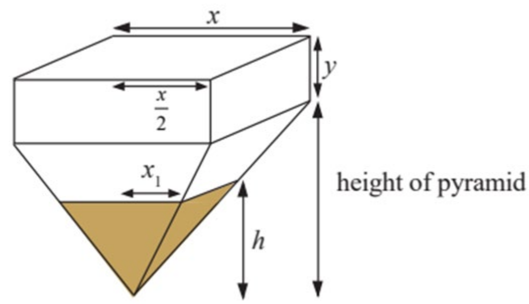
Refer to the diagram on the right.

By using similar triangles,

$$\frac{x_1}{\frac{x}{2}} = \frac{h}{\frac{1}{\sqrt{2}}x}$$

$$x_1 = \frac{\sqrt{2}}{2}h \text{ .... (1)}$$





Let V_1 be the volume of sand.

Since it is a small square base pyramid, its base area is $(2x_1)^2$.

$$V_1 = \frac{1}{3} \times (\text{square base pyramid}) \times (\text{level of sand})$$

$$= \frac{1}{3}(2x_1)^2 h \text{ .... (2)}$$

Substitute (1) into (2)

$$V_1 = \frac{1}{3}\left(2\left(\frac{\sqrt{2}}{2}h\right)\right)^2$$

$$V_1 = \frac{2}{3}h^3$$

Differentiate V_1 with respect to h

$$\frac{dV_1}{dh} = 2h^2$$

Using chain rule,

$$\frac{dh}{dt} = \frac{dV_1}{dt} \times \frac{d_1 h}{dV_1}$$

Given that sand is poured into the container at a constant rate of $0.1\,\text{cm}^3/\text{min}$, i.e. $\frac{dV_1}{dt} = 0.1$

$$\frac{dh}{dt} = (0.1)\left(\frac{1}{2h^2}\right)$$

$$= (0.1)\left(\frac{1}{2(0.05a)^2}\right)$$

$$= \frac{20}{a^2}$$

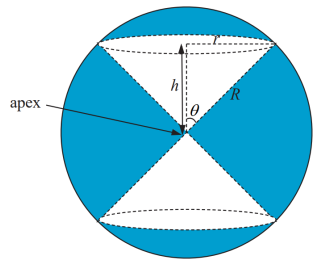
The rate of increase of the sand level when $h = 0.05a$ is $\frac{20}{a^2}$ m/mins.

17
Solution

(a) Use Pythagoras' Theorem

$r^2 + h^2 = R^2$

$$h = \sqrt{R^2 - r^2} \text{ .... (1)}$$



$$V = 2 \times \text{volume of two cones}$$

$$= 2 \times \frac{1}{3}\pi r^2 h \text{ .... (2)}$$

Substitute (1) into (2)

$$V = \frac{2}{3}\pi r^2 \sqrt{R^2 - r^2} \quad \triangleleft \text{ square both sides}$$

$$V^2 = \left(\frac{2}{3}\pi r^2 \sqrt{R^2 - r^2}\right)^2$$

$$= \frac{4}{9}\pi^2 r^4 \left(R^2 - r^2\right) \quad \text{(Shown) .... (3)}$$

(b) Differentiate (3) with respect to x

$$2V\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{4}{9}\pi^2\left[4r^3(R^2 - r^2) + r^4(-2r)\right]$$

$$= \frac{4}{9}\pi^2\left[4r^3R^2 - 4r^5 - 2r^5\right]$$

$$= \frac{4}{9}\pi^2\left[4r^3R^2 - 6r^5\right] \text{ .... (4)}$$

At stationary, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$

$$\frac{4}{9}\pi^2\left[4r^3R^2 - 6r^5\right] = 0$$

$$4r^3R^2 - 6r^5 = 0$$

$$r^3(4R^2 - 6r^2) = 0$$

$$4R^2 - 6r^2 = 0 \quad \text{or} \quad r^3 = 0$$

$$r^2 = \frac{2R^2}{3} \quad \text{or} \quad r = 0 \quad (\text{rejected } \because r \neq 0)$$

$$r = \pm\sqrt{\frac{2R^2}{3}}$$

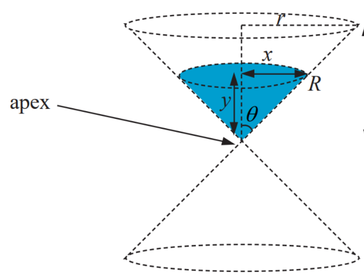
Since $r > 0$, $r = -\sqrt{\frac{2R^2}{3}}$ is rejected

$$\therefore \quad r = \sqrt{\frac{2}{3}}R \text{ .... (5)}$$

From (4): $$2V\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{4}{9}\pi^2\left[4r^3R^2-6r^5\right]$$
$$V\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{2}{9}\pi^2\left[4r^3R^2-6r^5\right]$$
Differentiate implicitly with respect to r,
$$V\frac{\mathrm{d}^2V}{\mathrm{d}r^2}+\left(\frac{\mathrm{d}V}{\mathrm{d}r}\right)^2=\frac{2}{9}\pi^2\left[12r^2R^2-30r^4\right]$$

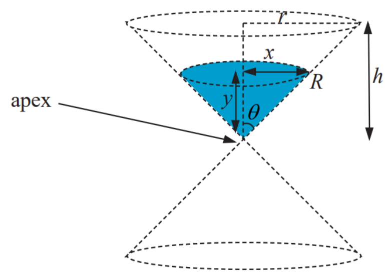
Substitute $r=\sqrt{\frac{2}{3}}R$ and $\frac{\mathrm{d}V}{\mathrm{d}r}=0$
$$V\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{2}{9}\pi^2\left[12\left(\frac{2}{3}R^2\right)R^2-30\left(\frac{2}{3}R^2\right)^2\right]$$
$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{2}{9}\pi^2\left[8R^4-\frac{40}{3}R^4\right]$$
$$=-\frac{32\pi^2R^4}{27V}<0 \quad \text{(max)}$$
Thus, V is maximum when $r=\sqrt{\frac{2}{3}}R.$

(c) From (1): $h=\sqrt{R^2-r^2}$
Substitute (5) into (1)
$$h=\sqrt{R^2-\frac{2}{3}R^2}$$
$$=\sqrt{\frac{R^2}{3}}$$
$$=\frac{R}{\sqrt{3}} \quad \text{................ (6)}$$



Refer to the triangle in the dagram
$$\tan\theta=\frac{r}{h}$$
$$\tan\theta=\frac{\sqrt{\frac{2}{3}}R}{\frac{R}{\sqrt{3}}}$$ ◃ replace r by $\sqrt{\frac{2}{3}}R$ from (5) and replace h by $\frac{R}{\sqrt{3}}$ from (6)
$$=\sqrt{2} \quad \text{(Shown)}$$

(a) From the diagram.



$$\frac{x}{y} = \tan\theta \quad \triangleleft \text{ obtain answer in } \textbf{(c)}$$

$$\frac{x}{y} = \sqrt{2}$$

$$y = \frac{x}{\sqrt{2}}$$

Let the volume of the sand flowing out be V_s.

$$V_s = \frac{1}{3}\pi x^2 y$$

$$= \frac{1}{3}\pi x^2\left(\frac{x}{\sqrt{2}}\right) \quad \triangleleft \text{ substitute } y = \frac{x}{\sqrt{2}}$$

$$= \frac{\pi x^3}{3\sqrt{2}}$$

Differentiate V_s with respect to x

$$\frac{\mathrm{d}V_s}{\mathrm{d}x} = \frac{\pi x^2}{\sqrt{2}}$$

Using Chain Rule,

$$\frac{\mathrm{d}V_s}{\mathrm{d}t} = \frac{\mathrm{d}V_s}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

Given that volume of sand flows out at a rate of 0.5 cm^3 per second, i.e. $\frac{\mathrm{d}V_s}{\mathrm{d}t} = -0.5$.

$$\therefore \quad -0.5 = \frac{\pi x^2}{\sqrt{2}} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-0.5\sqrt{2}}{\pi x^2}$$

When $x = 2$ cm,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-0.5\sqrt{2}}{\pi(2^2)}$$

$$= -0.0562697698$$

$$= -0.0563 \text{ (correct to 3 s.f.)}$$

The rate of decrease of radius of sand surface is 0.0563 cm/s.

18
Solution

(a) Given that the container holds k cm^3 of liquid when full,

$\therefore$ volume of a container $= k$

$$\frac{2}{3}\pi r^3 + \pi r^2 h = k$$

$$\pi r h = \frac{k}{r} - \frac{2}{3}\pi r^2$$

$$h = \frac{k}{\pi r^2} - \frac{2}{3}r \text{ ....(1)}$$

Let the cost of a can be C

$C = 6(2\pi r^2) + 9(2\pi r h) + 9(\pi r^2)$

$= 21\pi r^2 + 18\pi r h$ .... (2)

Substitute (1) into (2)

$$C = 21\pi r^2 + 18\pi r\left(\frac{k}{\pi r^2} - \frac{2}{3}r\right)$$

$$= 21\pi r^2 + \frac{18k}{r} - 12\pi r^2$$

$$= 9\pi r^2 + \frac{18k}{r}$$

Differentiate C with respect to x

$$\frac{\mathrm{d}C}{\mathrm{d}r} = 18\pi r - \frac{18k}{r^2}$$

At stationary, $\frac{\mathrm{d}C}{\mathrm{d}r} = 0$.

$$18\pi r - \frac{18k}{r^2} = 0$$

$$18\pi r = \frac{18k}{r^2}$$

$$r^3 = \frac{k}{\pi}$$

$$r = \sqrt[3]{\frac{k}{\pi}}$$

When $r = \sqrt[3]{\frac{k}{\pi}}$,

$$h = \frac{k}{\pi\left(\sqrt[3]{\frac{k}{\pi}}\right)^2} - \frac{2}{3}\left(\sqrt[3]{\frac{k}{\pi}}\right)$$

$$= \sqrt[3]{\frac{k}{\pi}} - \frac{2}{3}\left(\sqrt[3]{\frac{k}{\pi}}\right) = \frac{1}{3}\left(\sqrt[3]{\frac{k}{\pi}}\right)$$

The cheapest can is manufactured when $r = \sqrt[3]{\frac{k}{\pi}}$ and $h = \frac{1}{3}\left(\sqrt[3]{\frac{k}{\pi}}\right)$.

Differentiate $\frac{\mathrm{d}C}{\mathrm{d}r}$ with respect to x

$$\frac{\mathrm{d}^2C}{\mathrm{d}r^2} = 18\pi + \frac{36k}{r^3}$$

When $r = \sqrt[3]{\frac{k}{\pi}}$

$$\frac{\mathrm{d}^2C}{\mathrm{d}r^2} = 54\pi > 0$$

$\therefore$ r and h are minimum $r = \sqrt[3]{\frac{k}{\pi}}$.

(b) Volume of water in the container till cylindrical section $= \pi(5)^2 m$ m^3

Given that water is poured into the container till cylindrical section at the rate of ℓ litres per second.

At 72 seconds, the amount of water in the container till cylindrical section $= 72\ell$ litres

1000 litres $= 1$ m^3

$\therefore$ 72ℓ litres $= \frac{72\ell}{1000}$ m^3

Equating, $\frac{72\ell}{1000}$ m^3 $= \pi(5)^2 m$ m^3

$$m = \frac{9}{3125}\ell$$

(c) $\frac{\mathrm{d}V}{\mathrm{d}t} = \ell t$

$$V = \frac{\ell t^2}{2} + C$$

When $t = 0, V = 0, C = 0$

$$V = \frac{\ell t^2}{2}$$

When $V = 72\ell$,

$$72\ell = \frac{\ell t^2}{2}$$

$$t^2 = 144$$

$$t = 12 \text{ seconds } (\because t > 0)$$

(d) Volume of hemisphere $= \frac{2}{3}\pi(5^3)$

$$= \frac{250}{3}\pi$$

$$= 261.80$$

After 150 seconds, volume of water leaked $= 150(2) = 300 > 262.80$

$\therefore$ after 150 second, the amount of water is in the cylinder. Hence we only consider the rate of change of the height of water in the cylinder.

Let H be the height in cm of the water in the cylinder.

$$V = \pi r^2 H$$

When $r = 5$

$$V = \pi(5)^2 H$$

$$= 25\pi H \text{ .... (1)}$$

Differentiating (1) with respect to H

$$\frac{dV}{dH} = 25\pi$$

Using the Chain Rule,

$$\frac{dV}{dt} = \frac{dV}{dH} \times \frac{dH}{dt}$$

Water leaks from the container at a constant rate of 2 cm^3 per second, i.e. $\frac{dV}{dt} = -2$

When $\frac{dV}{dH} = 25\pi$ and $\frac{dV}{dt} = -2$

$$\therefore \quad -2 = 25\pi \times \frac{dH}{dt}$$

$$\frac{dH}{dt} = -\frac{2}{25\pi}$$

The height of the water in the container is decreasing at $\frac{2}{25\pi}$ cm per second, 150 seconds after the start of leaking.

19
Solution

(a) Given that intensity of the light, I, is inversely proportional to the square distance from the light source to a point,

i.e. $I \propto \frac{1}{x^2}$

$$\therefore \quad I = \frac{\sin\theta}{x^2}$$

$$\sin\theta = Ix^2 \quad \text{........................ (1)}$$

Using Pythagoras theorem,

$$PA^2 = OA^2 + OP^2$$
$$x^2 = R^2 + OP^2$$
$$OP = \sqrt{x^2 - R^2}$$

Using trigonometric ratio,

$$\sin\theta = \frac{OP}{OA}$$
$$\sin\theta = \frac{\sqrt{x^2 - R^2}}{x} \quad \text{........................ (2)}$$

Substitute (1) into (2).

$$Ix^2 = \frac{\sqrt{x^2 - R^2}}{x}$$
$$I = \frac{\sqrt{x^2 - R^2}}{x^3} \quad \text{(Shown) .... (3)}$$

(b) Let h denote the vertical height of the light source from the ground.

Using Pythagoras theorem,

$$\therefore \quad x^2 = R^2 + h^2 \quad \text{............................ (4)}$$

Substitute (4) into (3).

$$I = \frac{h}{\left(\sqrt{h^2 + R^2}\right)^3}$$
$$= \frac{h}{\left(h^2 + R^2\right)^{\frac{3}{2}}} \quad \text{............................ (5)}$$

Differentiate (5) with respect to h

$$\frac{\mathrm{d}I}{\mathrm{d}h} = \frac{\left(h^2 + R^2\right)^{\frac{3}{2}}(1) - \frac{3}{2}\left(h^2 + R^2\right)^{\frac{1}{2}}(2h)(h)}{\left(h^2 + R^2\right)^3} \quad < \text{use quotient rule}$$

$$= \frac{\left(h^2 + R^2\right)3h^2}{\left(h^2 + R^2\right)^{\frac{5}{2}}}$$

$$= \frac{R^2 - 2h^2}{\left(h^2 + R^2\right)^{\frac{5}{2}}}$$

At stationary, $\frac{\mathrm{d}I}{\mathrm{d}h}=0.$

$$\therefore \quad R^2-2h^2=0$$
$$h=\frac{R}{\sqrt{2}}$$

Hence, the light must be $\frac{\sqrt{2}}{2}R$ m above the centre of the circular area.

Use first derivative test to check if I is maximum:

h	$\frac{R}{\sqrt{2}}^{-}$	$\frac{R}{\sqrt{2}}$	$\frac{R}{\sqrt{2}}^{+}$
$\frac{\mathrm{d}I}{\mathrm{d}h}$	$\frac{R^2-2h^2}{(h^2+R^2)^{\frac{5}{2}}}>0$	0	$\frac{R^2-2h^2}{(h^2+R^2)^{\frac{5}{2}}}<0$
Slope	/	—	\

Hence, I is maximum when h is $\frac{\sqrt{2}}{2}R$ m.

(c) Using the Chain Rule,

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\mathrm{d}I}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$$

Given the light source is raised up at a constant speed of $0.5\ \mathrm{ms}^{-1}$, i.e. $\frac{\mathrm{d}h}{\mathrm{d}t}=0.5$

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{R^2-2h^2}{(h^2+R^2)^{\frac{5}{2}}}\times 0.5$$
$$=\frac{R^2-2h^2}{2(h^2+R^2)^{\frac{5}{2}}}$$

When $h=R$,

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{R^2-2R^2}{2(R^2+R^2)^{\frac{5}{2}}}$$
$$=\frac{-R^2}{2(2R^2)^{\frac{5}{2}}}$$
$$=\frac{-R^2}{2\left(2^{\frac{5}{2}}R^5\right)}$$
$$=\frac{-1}{8\sqrt{2}R^3}$$

The rate of change of the intensity of light at the circumference of the circular area when the light source is R m above the centre of the circular disc is $\frac{1}{8\sqrt{2}R^3}$ metres per second.

20
Solution

(a) Volume of the tank $= 2x^2y$

Given that volume of the tank is fixed at 36 000 cm^3,

$\therefore \quad 2x^2y = 36000$

$$y = \frac{18000}{x^2} \text{......................... (1)}$$

Let A be the total surface area of the tank

$$A = 6xy + 2x^2 \text{............................. (2)}$$

Substitute (1) into (2).

$$= 6x\left(\frac{18000}{x^2}\right) + 2x^2$$

$$= \frac{108000}{x} + 2x^2 \text{......................... (3)}$$

Differentiate (3) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = -\frac{108000}{x^2} + 4x \text{ .... (4)}$$

At stationary, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$$\therefore \quad \frac{108000}{x^2} = 4x$$

$$x^3 = \frac{108000}{4}$$

$$x = 30$$

Substitute $x = 30$ into (1).

$$y = \frac{18000}{30^2}$$

$$= 20$$

Differentiate (4) with respect to x

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \frac{216000}{x^3} + 4$$

When $x = 30$,

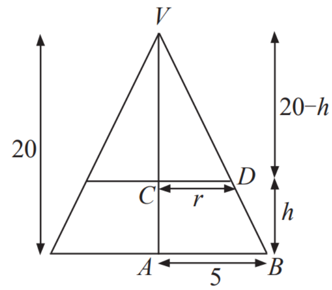
$$\left.\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right|_{x=30} = 12 > 0$$

$\therefore$ minimum value of A occurs at $x = 30$.

Therefore, the values of x and y are 30 cm and 20 cm respectively.

(b) Refer to the diagram.

ΔVAB and ΔVCD are similar triangles.



$$\frac{CD}{AB} = \frac{VC}{VA}$$

$$\frac{r}{5} = \frac{20-h}{20}$$

$$\therefore \quad r = \frac{20-h}{4} \quad \text{.........................} \quad (5)$$

(b) Volume of cone submerged in the tank

$$= \frac{1}{3}\pi(5)^2(20) - \frac{1}{3}\pi r(20-h) \quad \text{.........................} \quad (6)$$

Substitute (5) into (6).

$$= \frac{500}{3}\pi - \frac{1}{3}\pi\left(\frac{20-h}{4}\right)^2(20-h)$$

$$= \frac{500}{3}\pi - \frac{1}{48}\pi(20-h)^3$$

$$= \frac{\pi}{48}\left[8000-(20-h)^3\right]$$

Therefore, $V = 30\times 60\times h - \frac{\pi}{48}\left[8000-(20-h)^3\right]$

$$= 1800h - \frac{\pi}{48}\left[8000-(20-h)^3\right]$$

Differentiate V with respect to x

$$\frac{\mathrm{d}V}{\mathrm{d}h} = 1800 - \frac{\pi}{48}\left[3(20-h)^2\right]$$

$$= 1800 - \frac{\pi}{16}(20-h)^2$$

When $h = 10$,

$$\frac{\mathrm{d}V}{\mathrm{d}h} = 1800 - \frac{\pi}{16}(100)$$

Using chain rule

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

Given that $\frac{\mathrm{d}V}{\mathrm{d}t} = 1000$,

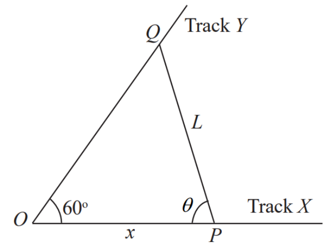
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{1800 - \frac{\pi}{16}(100)}\times 1000$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.562 \text{ (correct to 3 s.f.)}$$

The rate of increase of water level is 0.562 cm/s.

21
Solution

(a) Using sine rule,



$$\frac{x}{\sin(120^\circ-\theta)}=\frac{L}{\sin 60^\circ}$$

$$x=\frac{L}{\sin 60^\circ}\left[\sin(120^\circ-\theta)\right] \quad \triangleleft \text{ use addition formulae}$$

$$=\frac{L}{\sin 60^\circ}\left[\sin 120^\circ\cos\theta-\cos 120^\circ\sin\theta\right]$$

$$=\frac{2L}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta\right)$$

$$\therefore\ x=L\left(\cos\theta+\frac{1}{\sqrt{3}}\sin\theta\right) \quad \text{(Shown)} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

(b) Let t be time in seconds.

When $\theta=90^\circ$,

$$x=L\left(\cos 90^\circ+\frac{1}{\sqrt{3}}\sin 90^\circ\right)$$

$$=\frac{L}{\sqrt{3}}$$

When $\theta=30^\circ$,

$$x=L\left(\cos 30^\circ+\frac{1}{\sqrt{3}}\sin 30^\circ\right)$$

$$=L\left(\frac{\sqrt{3}}{2}+\frac{1}{2\sqrt{3}}\right)$$

$$=\frac{2L}{\sqrt{3}}$$

Given that x increases at a constant rate, i.e. $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{(\text{final } x)-(\text{initial } x)}{\text{time taken}}$.

$$\therefore \quad \frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\left(\frac{2L}{\sqrt{3}}-\frac{L}{\sqrt{3}}\right)}{60}$$

$$=\frac{L}{60\sqrt{3}}$$

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=L\left(-\sin\theta+\frac{1}{\sqrt{3}}\cos\theta\right)$$

Using Chain Rule

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}t}\ \text{....................}\ (2)$$

Substitute $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{L}{60\sqrt{3}}$ and $\frac{\mathrm{d}x}{\mathrm{d}\theta}=L\left(-\sin\theta+\frac{1}{\sqrt{3}}\cos\theta\right)$ into (2)

$$\frac{L}{60\sqrt{3}}=L\left(-\sin\theta+\frac{1}{\sqrt{3}}\cos\theta\right)\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

When $\theta = 60^\circ$,

$$\frac{L}{60\sqrt{3}}=L\left(-\sin 60^\circ+\frac{1}{\sqrt{3}}\cos 60^\circ\right)\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$\frac{1}{60\sqrt{3}}=\left(-\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}\times\frac{1}{2}\right)\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$\frac{1}{60\sqrt{3}}=\left(\frac{-3+1}{2\sqrt{3}}\right)\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{60}\ \text{rad s}^{-1}$$

The rate of change of θ at $\theta = 60°$ is $\frac{1}{60}$ rad s^{-1}.

(c) At initial, i.e. when $t = 0$ where $\theta = 90°$,

the distance from O to P, $x = L\left(\cos 90° + \frac{1}{\sqrt{3}}\sin 90°\right)$

$$= \frac{L}{\sqrt{3}}$$

Given that the rod starts to roll at a constant rate for 60 seconds, i.e.

at 1 second, P moves $\frac{L}{60\sqrt{3}}$ cm.

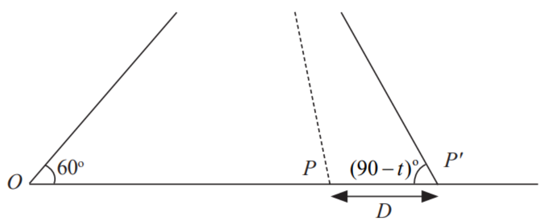
At t seconds $(0 \le t \le 60)$, P moves $\frac{L}{60\sqrt{3}}t$ cm.

$\therefore$ the distance from O the position of P at t seconds, $OP = \frac{L}{\sqrt{3}} + \frac{L}{60\sqrt{3}}t$

Given that θ is decreasing at a constant rate from 90° to 30° in one minute (60 seconds)

$\therefore$ at 1 second, θ is decreasing at 1°.

So, for t seconds, θ is decreasing at t°. Angle $QP'O = (90-t)^\circ$



$\therefore \quad OP' = L\left(\cos(90-t)^\circ + \frac{1}{\sqrt{3}}\sin(90-t)^\circ\right)$ ◁ complementary angles for trigonometry ratios $\cos(90-t)^\circ = \sin t^\circ$ and $\sin(90-t)^\circ = \cos t^\circ$

$$= L\left(\sin t^\circ + \frac{1}{\sqrt{3}}\cos t^\circ\right)$$

Thus,

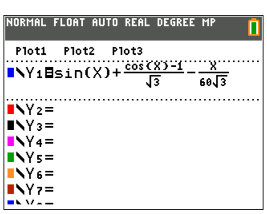
$D = OP' - OP$

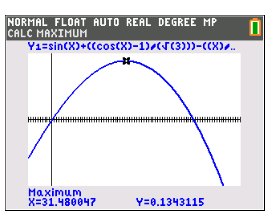
$= L\left(\sin t^\circ + \frac{1}{\sqrt{3}}\cos t^\circ\right) - \left(\frac{L}{\sqrt{3}} + \frac{L}{60\sqrt{3}}t\right)$

$= L\left(\sin t^\circ + \frac{\cos t^\circ - 1}{\sqrt{3}} - \frac{t}{60\sqrt{3}}\right)$

(d) $\frac{D}{L} = \sin t^\circ + \frac{\cos t^\circ - 1}{\sqrt{3}} - \frac{t}{60\sqrt{3}}$

Using GC,





The maximum value is 0.1343.

22
Solution

(a) At time t, $AQ = 20 - a$

Using trigonometric ratio

$$\tan \angle ARQ = \frac{20-a}{20}$$

At time t, $BQ = b$

Using trigonometric ratio

$$\tan \angle QRB = \frac{b}{20}$$

Using Addition Formulae: $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\begin{aligned}
\tan\theta &= \tan(\angle ARQ + \angle QRB) \\
&= \frac{\tan ARQ + \tan QRB}{1 - \tan ARQ \tan QRB} \\
&= \frac{\frac{20-a}{20} + \frac{b}{20}}{1 - \frac{20-a}{20}\left(\frac{b}{20}\right)} \\
&= \frac{20(20-a+b)}{400-20b+ab} \qquad \text{(Shown) .... (1)}
\end{aligned}$$

(b) Given $b = 2a$ .... (2)

Substitute (2) into (1)

$$\tan\theta = \frac{20(20+a)}{400-40a+2a^2}$$

Differentiate with respect to a,

$$\begin{aligned}
\sec^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}a} &= \frac{20[(400-40a+2a^2)-(-40+4a)(20+a)]}{(400-40a+2a^2)^2} \\
&= \frac{20(1200-80a-2a^2)}{(400-40a+2a^2)^2}
\end{aligned}$$

At stationary point, $\dfrac{\mathrm{d}\theta}{\mathrm{d}a} = 0$

$$\frac{20(1200-80a-2a^2)}{\sec^2\theta(400-40a+2a^2)^2} = 0$$

$$\cos^2\theta(1200-80a-2a^2) = 0$$

Since $\cos\theta \neq 0$ $\left(\theta \neq \dfrac{\pi}{2}\right)$.

$\therefore\ 1200 - 80a - 2a^2 = 0$

Using GC,

$a = 11.6$ (3 s.f.) or $a = -51.6$ (rejected since $a > 0$)

$\therefore\ \theta$ is maximum when $a = 11.6$

(c)(i) When $t = 0$, triangle PQR is an isosceles. $\therefore\ \theta = 45°$.

Substitute $\theta = 45°$ into (1).

$$\frac{20(20-a+b)}{400-20b+ab} = \tan 45°$$

$$\frac{20(20-a+b)}{400-20b+ab} = 1$$

$$400-20a+20b = 400-20b+ab$$

$$b = \frac{20a}{40-a} \quad \text{(Shown)}$$

(c)(ii) From **(c)(i)**

$$b = \frac{20a}{40-a}$$

Differentiate b with respect to t

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{800}{(40-a)^2}\frac{\mathrm{d}a}{\mathrm{d}t}$$

Given that A moves at a constant rate of 0.5 ms^{-1}, i.e. for 1 second, a moves at 0.5 m.

At $t = 30$, $a = 0.5\times 30 = 15$ m

Also given that $\frac{\mathrm{d}a}{\mathrm{d}t} = 0.5$

Substitute $\frac{\mathrm{d}a}{\mathrm{d}t} = 0.5$ and $a = 15$ into $\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{800}{(40-a)^2}\frac{\mathrm{d}a}{\mathrm{d}t}$

$$\therefore \quad \frac{\mathrm{d}b}{\mathrm{d}t} = \frac{800}{(40-15)^2}(0.5)$$

$$= 0.64\ \mathrm{ms}^{-1} \quad \text{(correct to 2 dp)}$$

The speed of B at $t = 30$ is 0.64 ms^{-1}.

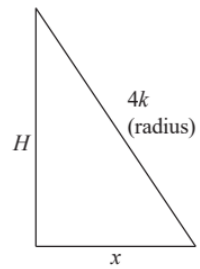
23
Solution

(a) Let H be the vertical height of the trapezium.

By Pythagoras Theorem,

$$H^2 + x^2 = (4k)^2$$

$$H = \sqrt{(4k)^2 - x^2} \quad \text{.................. (1)}$$



$$S = \frac{1}{2}(2x+8k)H \quad \text{............. (2)}$$ ◁ Area of trapezium *ABCD*

Substitute (1) into (2)

$$= \frac{1}{2}(2x+8k)\sqrt{(4k)^2 - x^2}$$

$$= (x+4k)\sqrt{16k^2 - x^2} \quad \text{(Shown)}$$

(b) Differentiate S with respect to x

$$\frac{\mathrm{d}S}{\mathrm{d}x} = \sqrt{16k^2 - x^2} - (x+4k)x(16k^2 - x^2)^{-\frac{1}{2}}$$

At stationary, $\frac{\mathrm{d}S}{\mathrm{d}x} = 0$.

$$\therefore \quad \frac{-2x^2 - 4kx + 16k^2}{\sqrt{16k^2 - x^2}} = 0$$

$$x^2 + 2kx - 8k^2 = 0$$

$$(x-2k)(x+4k) = 0$$

$$x = 2k \text{ or } x = -4k \quad \text{(Rejected since } x > 0\text{)}$$

From (3): $$\frac{\mathrm{d}S}{\mathrm{d}x} = \frac{-2x^2 - 4kx + 16k^2}{\sqrt{16k^2 - x^2}}$$

$$= \frac{-2(x^2 + 2kx - 8k^2)}{\sqrt{16k^2 - x^2}}$$

Differentiate $\frac{\mathrm{d}S}{\mathrm{d}x}$ with respect to x

$$\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = -2\left[\frac{\sqrt{16k^2 - x^2}(2x+2k) - (x^2 + 2kx - 8k^2)\frac{1}{2}(16k^2 - x^2)^{-\frac{1}{2}}(-2x)}{16k^2 - x^2}\right]$$

When $x = 2k$

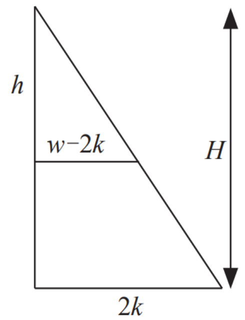
$$\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = -2\frac{\sqrt{12k^2}\,6k}{12k^2} = -\sqrt{12} < 0$$

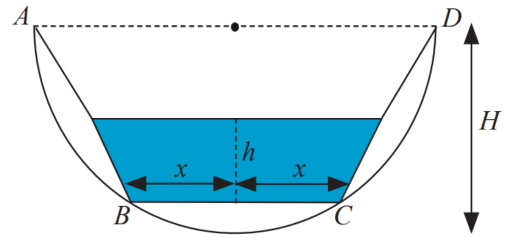
Area of trapezium is maximum when $x = 2k$.

(c) Given $x = 2k$, substitute $x = 2k$ into (1)

$$H = \sqrt{(4k)^2 - (2k)^2} \qquad \triangleleft\ H \text{ be the vertical height of the trapezium}$$
$$= \sqrt{12k^2}$$
$$= 2\sqrt{3}k \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

Using similar triangles :





$$\frac{h}{H} = \frac{w-x}{x} \qquad \triangleleft \text{ Given } x = 2k$$
$$\frac{h}{H} = \frac{w-2k}{2k}$$
$$\frac{h2k}{H} = w - 2k$$
$$w = 2k + \frac{h2k}{H} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4)$$

Substitute (3) into (4)

$$w = 2k + \frac{h2k}{(2\sqrt{3}k)}$$
$$= 2k + \frac{h}{\sqrt{3}}$$

$$\therefore \quad V = (\text{area of trapezium}) \times (\text{length of the bucket})$$
$$= \left[\frac{1}{2} \times \text{height of trapezium} \times (\text{sum of its parallel sides})\right] \times 3$$
$$= \left[\frac{1}{2} \times h \times (4k + 2w)\right] \times 3$$
$$= \frac{3}{2} \times h \times 2(2k + w) \qquad \triangleleft \text{ substitute } w = 2k + \frac{h}{\sqrt{3}}$$
$$= \frac{3}{2}h\left(4k + 2\left(\frac{h}{\sqrt{3}} + 2k\right)\right)$$
$$= 3h\left(4k + \frac{h}{\sqrt{3}}\right) \qquad \text{(Shown)}$$

(d) Differentiate V with respect to h

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \left(4k + \frac{h}{\sqrt{3}}\right)3 + 3h\left(\frac{1}{\sqrt{3}}\right)$$
$$= 12k + \frac{6h}{\sqrt{3}}$$

When $h = \sqrt{3}k$,

$$\frac{\mathrm{d}V}{\mathrm{d}h} = 18k$$

Using Chain Rule,

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$
$$=\frac{1}{18k}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

Given that the bucket is filled with water at a constant rate of 0.2 m^3/s, i.e. $\frac{\mathrm{d}V}{\mathrm{d}t}=0.2$.

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{18k}\times 0.2$$
$$=\frac{1}{90k}\ \mathrm{m/s}$$

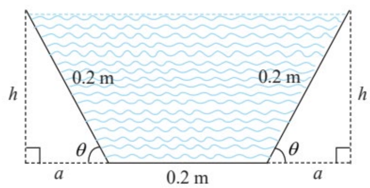
The rate at which the depth is increasing at the instant when the depth is $k\sqrt{3}$ m is $\frac{1}{90k}$ m/s.

Exercise 9

E Higher Order Questions

24
Solution

(a) Let h and a be the height and base of the right angle triangle formed respectively as shown.



Refer to the above diagram.
Using trigonometric ratio,

$$h = 0.2\sin\theta$$
$$a = 0.2\cos\theta$$

Let V be volume of the tray

$$\begin{aligned}\therefore\ V &= \text{Area of base trapezium} \times \text{length}\\ &= \frac{1}{2}\left[0.2 + (0.2 + 2a)\right](h) \times 2\\ &= (0.4 + 0.4\cos\theta)(0.2\sin\theta)\\ &= 0.08(\sin\theta + \sin\theta\cos\theta)\\ &= 0.04(2\sin\theta + \sin 2\theta) \quad \text{(Shown)} \text{ .... (1)}\end{aligned}$$

(b) Differentiate (1) with respect to θ

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\theta} &= 0.04(2\cos\theta + 2\cos 2\theta)\\ &= 0.08(\cos\theta + \cos 2\theta)\end{aligned}$$

When the capacity of the tray can hold as large as possible, i.e. $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$.

$$\begin{aligned}\therefore \quad \cos\theta + \cos 2\theta &= 0\\ \cos\theta + (2\cos^2\theta - 1) &= 0\\ 2\cos^2\theta + \cos\theta - 1 &= 0\\ (2\cos\theta - 1)(\cos\theta + 1) &= 0\end{aligned}$$

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -1 \quad (\text{rejected, since } \theta \text{ is acute})$$

$$\theta = \frac{\pi}{3}$$

Method 1 : (2nd Derivative Test)

$$\frac{d^2V}{d\theta^2} = -0.08(\sin\theta + 2\sin 2\theta)$$

Since θ is acute, $\sin\theta > 0$ and $\sin 2\theta > 0$

$$\therefore \frac{d^2V}{d\theta^2} = -0.08(\sin\theta + 2\sin 2\theta) < 0$$

Hence V is maximum when $\theta = \frac{\pi}{3}$.

Method 2 : (1st Derivative Test)

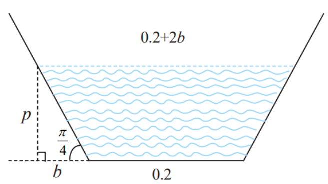
θ	$\left(\frac{\pi}{3}\right)^-$	$\frac{\pi}{3}$	$\left(\frac{\pi}{3}\right)^+$
$\frac{dV}{d\theta} = 0.08(\cos\theta + \cos 2\theta)$	+	0	–
	/	—	\

Hence V is maximum when $\theta = \frac{\pi}{3}$.

(c) Let p and b be depth and base respectively of right angle triangle formed with the side wall of the tray.
From the diagram,

$$\tan\frac{\pi}{4} = \frac{p}{b}$$

$$\therefore \quad p = b$$



Let V_p be volume of oil when depth is p.

$$V_p = (\text{Area of cross-section}) \times \text{height}$$

$$= (\text{Area of trapezium}) \times \text{height}$$

$$= \frac{1}{2}\left[0.2 + (0.2 + 2b)\right] p \times 2$$

$$= (0.4 + 2p)p \quad \triangleleft \text{ since } b = p$$

$$= 0.4p + 2p^2 \text{ .... } (2)$$

Differentiate (2) with respect to θ

$$\frac{dV_p}{dp} = 0.4 + 4p$$

Using the Chain Rule,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}V_p} \times \frac{\mathrm{d}V_p}{\mathrm{d}t}$$

Given that oil is poured at a constant rate of 0.02 m^3 into the tray every second, i.e. $\frac{\mathrm{d}V_p}{\mathrm{d}t} = 0.02$

$$= \frac{1}{0.4+4p} \times 0.02$$

When $p = 0.1$,

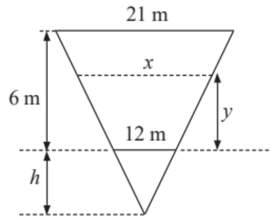
$$= \frac{1}{0.4+4(0.1)} \times 0.02$$

$$= \frac{1}{0.8} \times 0.002$$

$$= 0.0025 \text{ ms}^{-1} \quad \text{(Shown)}$$

25
Solution

(a) By similar triangles,



$$\frac{h+6}{h} = \frac{21}{12}$$
$$12h + 72 = 21h$$
$$h = 8$$

By similar triangles,

$$\frac{x}{12} = \frac{y+h}{h}$$
$$\frac{x}{12} = \frac{y+8}{8}$$
$$x = \frac{3y+24}{2} \quad \text{...............................} \ (1)$$

Volume of the water in the gutter, V = (Area of trapezium) $\times$ Length of the gutter

$$= \frac{y}{2}(x+12)(75) \quad \text{...............................} \ (2)$$

Substitute (1) into (2)

$$V = \frac{y}{2}\left(\frac{3y+24}{2} + 12\right)(75)$$
$$= \frac{225y^2}{4} + 900y \quad \text{(Shown)} \ \text{...............................} \ (3)$$

(b) When $t = 0.5$,

Amount of water flows into the gutter for 0.5 hours $= \int_0^{\frac{1}{2}} 3\times 10^6 t^{-\frac{3}{2}} e^{\left(\frac{3}{2}-\frac{6}{t}\right)} \, dt$
$= 9.3724$

Total amount of water in the gutter for 0.5 hours = Volume of the water in the gutter

$$7112.25 + 9.3724 = \frac{225y^2}{4} + 900y$$
$$y = 5.806$$

$\therefore$ the depth of water the gutter at 0.5 hour is $y = 5.806$

Differentiate (2) with respect to y

$$\frac{dV}{dy} = \frac{225y}{2} + 900$$
$$= \frac{2}{225y + 1800}$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \left(\frac{2}{225y+1800}\right) \times 3\times10^{6} t^{-\frac{3}{2}} e^{\left(\frac{3}{2}-\frac{6}{t}\right)}$$

When $t = 0.5$ and $y = 5.806$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.150437 \text{ m/h}$$

$$= 0.150 \text{ m/h} \ \text{(correct to 3 sf)}$$

The rate of change of y at the instant where water is pouring into the gutter for half hour is 0.150 m/h.

(c) When $y = 6, V = 7425$

When $y = 4, V = 4500$

When $y = 2, V = 2025$

Total time taken for the pumps to drain out the water completely

$$= \frac{\text{volume of water in time interval}}{(\text{no. of pump}) \times \text{rate of pump water}}$$

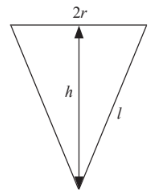
$= (\text{Time taken when } 4 \le y < 6) + (\text{Time taken when } 2 \le y < 4) + (\text{Time taken when } 0 < y < 2)$

$$= \frac{7425-4500}{3\times85} + \frac{4500-2025}{2\times85} + \frac{2025}{85}$$

$= 49.85$ min

26
Solution

(a) Using Pythagoras theorem



$$r^2 + h^2 = 1$$
$$h = \sqrt{1-r^2} \quad (h > 0) \text{ .... (1)}$$

Let V be the volume of the cone

$$V = \frac{1}{3}\pi r^2 h \text{ .... (2)}$$

Substitute (1) into (2)

$$V = \frac{1}{3}\pi r^2 \sqrt{1-r^2}$$ ◁ apply product rule

Differentiate V respect to r

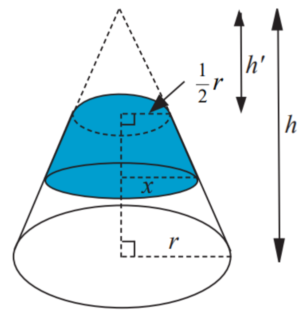
$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{2}{3}\pi r\sqrt{1-r^2} - \frac{\pi r^3}{3\sqrt{1-r^2}}$$

At maximum, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$

$$\frac{2}{3}\pi r\sqrt{1-r^2} - \frac{\pi r^3}{3\sqrt{1-r^2}} = 0$$
$$\frac{2}{3}\pi r(1-r^2) - \frac{1}{3}\pi r^3 = 0$$
$$2(1-r^2) - r^2 = 0$$
$$3r^2 = 2$$
$$r = \sqrt{\frac{3}{2}} \quad (\because r > 0)$$

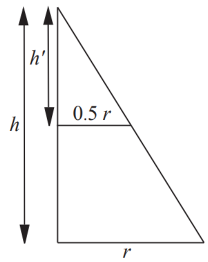
V is maximum when $r = \sqrt{\frac{3}{2}}$.

(b) Given that $r=\sqrt{\frac{2}{3}}$, substitute (1)



$$h=\sqrt{1-\frac{2}{3}}$$
$$=\frac{1}{\sqrt{3}}$$

Using similar triangle,

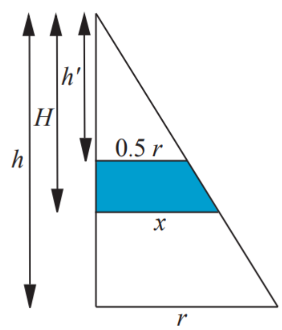


$$\frac{h'}{h}=\frac{0.5r}{r}$$
$$\frac{h'}{h}=0.5$$
$$h'=\frac{1}{2}h$$

when $h=\frac{1}{\sqrt{3}}$

$$h'=\frac{1}{2\sqrt{3}}$$

Let $H-h'$ be depth of liquid at time t minutes.



$$\frac{x}{r}=\frac{H}{h}$$
$$H=\frac{x}{r}\times h$$

When $r=\sqrt{\frac{2}{3}}$ and $h=\frac{1}{\sqrt{3}}$

$$H=\frac{x}{\sqrt{2}}$$

Let the volume of the liquid in the nozzle at time t minutes be V

$$V=\frac{1}{3}\pi x^2H-\frac{1}{3}\pi(0.5r)^2h' \quad \text{< volume of cone formulae} =\frac{1}{3}\pi r^2h$$

$$V=\frac{1}{3}\pi x^2H-\frac{1}{3}\pi\left(\frac{1}{2}\sqrt{\frac{2}{3}}\right)^2\left(\frac{1}{2\sqrt{3}}\right)$$

$$=\frac{1}{3}\pi\left[x^2H-\left(\frac{1}{2}\sqrt{\frac{2}{3}}\right)^2\left(\frac{1}{2\sqrt{3}}\right)\right]$$

$$=\frac{1}{3}\pi\left[\frac{x^3}{\sqrt{2}}-\frac{1}{12\sqrt{3}}\right] \text{ .... (3) (Shown)}$$

Differentiate V with respect to x

$$\frac{\mathrm{d}V}{\mathrm{d}x}=\frac{x^2\pi}{\sqrt{2}}$$

Given that the volume of the liquid in the nozzle is 0.2 m^3, substitute $V=0.2$ into (3)

$$0.2=\frac{1}{3}\pi\left[\frac{x^3}{\sqrt{2}}-\frac{1}{12\sqrt{3}}\right]$$

$$x=0.69667558$$

Using Chain Rule,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$$

Given that net inflow of liquid is $0.05 - 0.03 = 0.02$ m³ per minute, i.e. $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.02$

$$= \frac{\sqrt{2}}{x^2\pi} \times 0.02$$

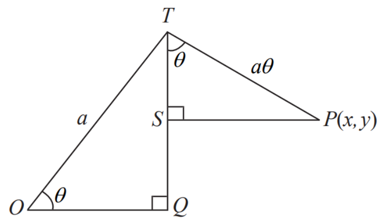
When $x = 0.69667558$

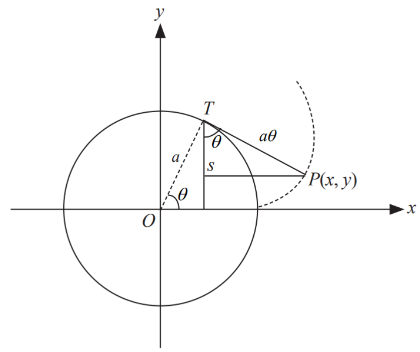
$$= \frac{\sqrt{2}}{\pi(0.69667558)^2} \times (0.02)$$
$$= 0.0185$$

$\therefore$ x increases at 0.0185 m per minute

27
Solution

(a) Refer to the diagram (triangle OTQ)





$$\cos\theta = \frac{OQ}{a}$$
$$OQ = a\cos\theta$$
$$\sin\theta = \frac{TQ}{a}$$
$$TQ = a\sin\theta$$

Arc length of circle $= a\theta$

$\therefore\ TP = a\theta$

Refer to the diagram (triangle PTS)

$SP = a\theta\sin\theta,\ TS = a\theta\cos\theta$

$x = OQ + SP$

$= a\cos\theta + a\theta\sin\theta$ (shown)

$y = TQ - TS$

$= a\sin\theta - a\theta\cos\theta$ (shown)

$\therefore$ the parametric equations are $x = a(\cos\theta + \theta\sin\theta),\quad y = a(\sin\theta - \cos\theta),\quad$ for $0 \le \theta \le \frac{\pi}{2}$. (Shown)

(b) Given $x = a(\cos\theta + \theta\sin\theta)$ .... (1)

and $y = a(\sin\theta - \cos\theta)$ .... (2)

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta$$
$$= a\theta\cos\theta$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta$$
$$= a\theta\sin\theta$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a\theta\sin\theta}{a\theta\cos\theta}$$
$$= \tan\theta \text{ .... (3)}$$

When $\theta = \frac{\pi}{3}$, substitute $\theta = \frac{\pi}{3}$ into (1), (2) and (3).

From (1) $x = a\left(\cos\frac{\pi}{3} + \frac{\pi}{3}\sin\frac{\pi}{3}\right)$

$$= a\left(\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)$$

From (2) $y = a\left(\sin\frac{\pi}{3} - \frac{\pi}{3}\cos\frac{\pi}{3}\right)$

$$= a\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$$

From (3) $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\frac{\pi}{3}$

$$= \sqrt{3}$$

$\therefore$ Gradient of normal $= -\frac{1}{\sqrt{3}}$

Equation of normal at W is

$$y - a\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}\left(x - \frac{a}{2} - \frac{\pi a\sqrt{3}}{6}\right)$$

$$\sqrt{3}y - \sqrt{3}a\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi\sqrt{3}a}{6} = -x + \frac{a}{2} + \frac{\pi\sqrt{3}a}{6}$$

$$\sqrt{3}y = \frac{a}{2} + \frac{3a}{2} - x$$

$$\sqrt{3}y = 2a - x \qquad \text{(Shown)}$$

(c) Given that at $\theta = \frac{\pi}{3}$, x increases at a rate of 0.3 units per second, i.e. $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.3$.

Using Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= \left(\tan\frac{\pi}{3}\right)(0.3)$$

$$= \frac{3\sqrt{3}}{10}$$

Given $z = xy$

Differentiate z with respect to t

$$\frac{\mathrm{d}z}{\mathrm{d}t} = x\frac{\mathrm{d}y}{\mathrm{d}t} + y\frac{\mathrm{d}x}{\mathrm{d}t}$$

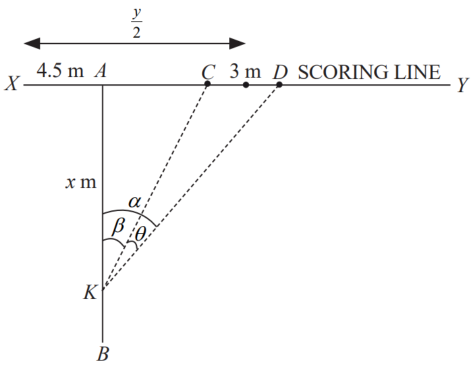
$$= a\left(\frac{1}{2} + \frac{\pi\sqrt{3}}{6}\right)\left(\frac{3\sqrt{3}}{10}\right) + a\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)(0.3)$$

$$= 0.834a \text{ (3sf)}$$

$\therefore$ the rate of change of z at W is $0.834a$ units per second.

28
Solution

(a)



Let angle AKD be α.

$$\tan AKD = \frac{\frac{y}{2} - 4.5 + 1.5}{x}$$

$$\tan \alpha = \frac{\frac{y}{2} - 4.5 + 1.5}{x}$$

$$= \frac{y-6}{2x}$$

Let angle AKD be β.

$$\tan AKD = \frac{\frac{y}{2} - 4.5 - 1.5}{x}$$

$$\tan\beta = \frac{\frac{y}{2} - 4.5 - 1.5}{x}$$

$$= \frac{y-12}{2x}$$

From the diagram.

$$\theta = \alpha - \beta$$

$$\tan\theta = \tan(\alpha - \beta)$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{\frac{y-6}{2x} - \frac{y-12}{2x}}{1 + \left(\frac{y-6}{2x}\right)\left(\frac{y-12}{2x}\right)}$$

$$= \frac{12x}{4x^2 + (y-6)(y-12)}$$

$$= \frac{12x}{4x^2 + A}$$ (Shown) .... (1), where $A = (y-6)(y-12)$

(b) Differentiate (1) with respect to x

$$\sec^2\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = \frac{12(4x^2 + A) - 12x(8x)}{(4x^2 + A)^2}$$

$$\sec^2\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = \frac{12(A - 4x^2)}{(4x^2 + A)^2}$$

$$\sec^2\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = \frac{12(\sqrt{A} - 2x)(\sqrt{A} + 2x)}{(4x^2 + A)^2}$$

To maximises $\tan\theta$, $\dfrac{d\theta}{dx}=0$.

$$\sec^2\theta(0)=\frac{12(\sqrt{A}-2x)(\sqrt{A}+2x)}{\left[4x^2+A\right]^2}$$

$$0=\frac{12(\sqrt{A}-2x)(\sqrt{A}+2x)}{\left[4x^2+A\right]^2}$$

$$0=(\sqrt{A}-2x)(\sqrt{A}+2x)$$

$$\therefore \quad x=\frac{\sqrt{A}}{2} \quad \text{or} \quad x=-\frac{\sqrt{A}}{2} \quad \text{(Rejected, since } x>0\text{)}$$

Thus for maximum θ, the player should be $\dfrac{\sqrt{A}}{2}$ m away from the scoreline.

(c) Given that the length of the scoring line XY is 20 m, i.e. $y=20$.
Substitute $y=20$ into $A=(y-6)(y-12)$.
$\therefore A=112$

Substitute $A=112$ into $x=\dfrac{\sqrt{A}}{2}$

$$\therefore \quad \frac{\sqrt{112}}{2}=2\sqrt{7}$$

The player should be $2\sqrt{7}$ m away from the scoreline.

(d) When the player is 15 m away from the goal line, i.e. $x=15$.
Substitute $x=15$ and $A=112$ into (1).

$$\therefore \quad \tan\theta=\frac{45}{253}$$

Given that the rugby player runs at a constant speed of 50 m per minute, $\dfrac{dx}{dt}=-50$.

Using Chain Rule

$$\begin{aligned}\frac{d\theta}{dt}&=\frac{d\theta}{dx}\times\frac{dx}{dt}\\&=\frac{12(\sqrt{A}-2x)(\sqrt{A}+2x)}{\sec^2\theta(4x^2+A)^2}\times -50\\&=\frac{12(A-4x^2)}{(1+\tan^2\theta)(4x^2+A)^2}\times -50\end{aligned}$$

When $x=15$, $\tan\theta=\dfrac{45}{253}$, $A=112$

$$\begin{aligned}&=\frac{12(112)-48(15)^2}{\left[1+\left(\frac{45}{253}\right)^2\right]\left[4(15)^2+112\right]^2}\times -50\\&=0.447 \text{ rad/min (to 3 sf)}\end{aligned}$$

The rate of change of θ at the instant visual angle when the player is 15 m away from the goal line is 0.447 rad/min.

29
Solution

(a) Given $x = r(\theta - \sin\theta)$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = r(1-\cos\theta) \text{ .... (1)}$$

Since $-1 \le \cos\theta \le 1$ for all $\theta \ge 0$,

$$0 \le 1-\cos\theta \le 2$$
$$0 \le r(1-\cos\theta) \le 2r$$

$\therefore \dfrac{\mathrm{d}x}{\mathrm{d}\theta} \ge 0$ for all $\theta \ge 0$. (Shown)

(b) Given $y = r(1-\cos\theta)$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = r\sin\theta \text{ .... (2)}$$

When $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 0$.

From (1): $r(1-\cos\theta) = 0$

$\cos\theta = 1$

$\theta = 0, 2\pi, 4\pi, \ldots$

When $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 0$

From (2): $r\sin\theta = 0$

$\sin\theta = 0$

$\theta = 0, \pi, 2\pi, 3\pi, \ldots$

Therefore, for both $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 0$

$\theta = 0, 2\pi, 4\pi, \ldots$

$\therefore\ \theta = 2n\pi$, where $n = 0, 1, 2, \ldots$

(c) The wheel rotates at a constant rate at k rad/s, i.e. $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = k$

Since point P is at rest at least once every 0.5 seconds, the wheel rotates at least 2π every 0.5 seconds.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \ge \frac{2\pi}{0.5}$$
$$k \ge 4\pi$$

$\therefore$ the range of values of k if the point P is at rest at least once every 0.5 s is $k \ge 4\pi$.

(d) The wheel rotates at a constant rate at k rad/s.

Given that $k = 8\pi$, the wheel rotates at 8π rad per second.

When $t = \frac{8}{3}$, $\theta = \frac{8}{3}(8\pi)$

$$= \frac{64}{3}\pi$$

Using Chain Rule,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$$
$$= 2(1 - \cos\theta)(8\pi)$$

When $t = \frac{8}{3}\pi, \theta = \frac{64}{3}\pi$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 16\pi\left(1 - \cos\left(\frac{64}{3}\pi\right)\right)$$
$$= 24\pi$$

The exact value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $t = \frac{8}{3}$ is 24π rad per second.

(e) From the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$$
$$= (2\sin\theta)(8\pi)$$
$$= 16\pi\sin\theta$$

Since $-1 \le \sin\theta \le 1$,

$\therefore \quad -16\pi \le 16\pi\sin\theta \le 16\pi$

$\therefore$ the maximum value of $\frac{\mathrm{d}y}{\mathrm{d}t}$ is 16π.

Exercise 9

F Exam Style Questions

30
Solution

(a) $3x^2 - 2xy + 5y^2 = 14$ .... (1)

Differentiate (1) with respect to x

$$6x - 2x\frac{dy}{dx} - 2y + 10y\frac{dy}{dx} = 0$$

$$(2x - 10y)\frac{dy}{dx} = 6x - 2y$$

$$\frac{dy}{dx} = \frac{3x - y}{x - 5y} \quad \text{(Shown)}$$

(b) Given that the normal is parallel to the x-axis, i.e. $\frac{dy}{dx} = 0$.

$$\frac{3x - y}{x - 5y} = 0$$

$$x - 5y = 0$$

$$y = 0.2x$$

Substitute $y = 0.2x$ into (1)

$$3x^2 - 2x(0.2x) + 5(0.2x)^2 = 14$$

$$2.8x^2 = 14$$

$$x = \pm\sqrt{5}$$

The exact x-coordinates are $\pm\sqrt{5}$.

(c) Given $y = 1$, substitute $y = 1$ into (1)

$$3x^2 - 2x(1) + 5(1)^2 = 14$$

$$3x^2 - 2x - 9 = 0$$

Using GC, $x = -1.4305$ or $x = 2.0972$

(c) Using Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Given that y decreases at a constant rate of 7 units per second, i.e. $\frac{dy}{dt} = -7$.

$$-7 = \left(\frac{3x - 1}{x - 5}\right)\left(\frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = \frac{7(5 - x)}{3x - 1}$$

When $x = 2.0972$,

$$\frac{dx}{dt} = 3.84 \text{ units per second (3 s.f.)}$$

The corresponding rate of change in x when $y = 1$ is 3.84 units per second.

31
Solution

(a) Given $\quad x=\sin^2\theta$ .... (1)

and $\quad y=\cos^3\theta$ .... (2)

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=2\sin\theta\cos\theta$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=-3\cos^2\theta\sin\theta$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$=\frac{-3\cos^2\theta\sin\theta}{2\sin\theta\cos\theta}$$

$$=-\frac{3}{2}\cos\theta$$

Given that the point P has parameter p, i.e. $\theta = p$.

Substitute $\theta = p$ into (1) and (2)

From (1): $x=\sin^2 p$

From (2): $y=\cos^3 p$

$\therefore\ P(\sin^2 p,\ \cos^3 p)$

Equation of tangent at $(\sin^2 p, \cos^3 p)$

$$y-\cos^3 p=\left(-\frac{3}{2}\cos p\right)(x-\sin^2 p)$$

$$y=\left(-\frac{3}{2}\cos p\right)x+\frac{3}{2}\cos p\sin^2 p+\cos^3 p \quad \text{(Shown)}$$

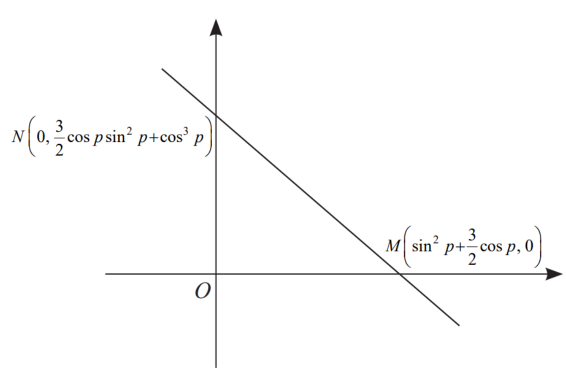
(b) When $y=0$, $\left(\frac{3}{2}\cos p\right)x=\frac{3}{2}\cos p\sin^2 p+\cos^3 p$

$$x=\sin^2 p+\frac{3}{2}\cos^2 p$$

$\therefore$ the coordinates of $M=\left(\sin^2 p+\frac{3}{2}\cos^2 p,\ 0\right)$

When $x=0$, $y=\frac{3}{2}\cos p\sin^2 p+\cos^3 p$

$\therefore$ the coordinates of $N=\left(0,\ \frac{3}{2}\cos p\sin^2 p+\cos^3 p\right)$



Area of ΔOMN

$$=\frac{1}{2}\left(\frac{3}{2}\cos p\sin^2 p+\cos^3 p\right)\left(\sin^2 p+\frac{3}{2}\cos^2 p\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\right)(\cos p)(3\sin^2 p+2\cos^2 p)\left(\frac{1}{3}\right)(3\sin^2 p+2\cos^2 p)$$

$$=\frac{1}{12}(\cos p)(3\sin^2 p+2\cos^2 p)^2$$

$$=\frac{1}{12}(\cos p)(\sin^2 p+2\sin^2 p+2\cos^2 p)^2$$

$$=\frac{1}{12}(\cos p)(\sin^2 p+2)^2 \quad \text{(Shown)}$$

(c) Let A be the area of ΔOMN.

$$\therefore \quad A=\frac{1}{12}(\cos p)(\sin^2 p+2)^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}p}=\frac{1}{12}(\cos p)(2)(\sin^2 p+2)(2\sin p\cos p)-\frac{1}{12}(\sin p)(\sin^2 p+2)^2$$

$$=\frac{1}{12}(\sin p)(\sin^2 p+2)\left[4\cos^2 p-(\sin^2 p+2)\right] \text{ .... (3)}$$

Given $x=\frac{3}{4}$, substitute into (1).

$$\sin^2 p=\frac{3}{4}$$

$$p=\frac{\pi}{3} \quad \text{or} \quad p=-\frac{\pi}{3} \text{ (rejected as } 0\le p<\frac{1}{2}\pi)$$

Substitute $p=\frac{\pi}{3}$ into (3)

$$\frac{\mathrm{d}A}{\mathrm{d}p}=\frac{1}{12}\left(\sin\frac{\pi}{3}\right)\left(\sin^2\frac{\pi}{3}+2\right)\left[4\cos^2\frac{\pi}{3}-\left(\sin^2\frac{\pi}{3}+2)\right)\right]$$

$$=\frac{1}{12}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{11}{4}\right)\left(1-\frac{11}{4}\right)$$

$$=-\frac{77\sqrt{3}}{384}$$

Using Chain Rule

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}p}\times\frac{\mathrm{d}p}{\mathrm{d}t}$$

Given that p increases at a constant rate of 0.2 radians per second, $\frac{\mathrm{d}p}{\mathrm{d}t}=0.2$

$$=-\frac{77\sqrt{3}}{384}\times 0.2$$

$$=-\frac{77\sqrt{3}}{1920}\text{ units}^2/\text{s}$$

The rate of the area of triangle OMN decreasing when $x=\frac{3}{4}$ is $\frac{77\sqrt{3}}{1920}$ units per second.

32
Solution

Let V and A be the volume and surface area of the spherical mothball respectively.
Given that the spherical mothball sublimes; its volume, decreases at a rate, proportional to its surface area,

i.e. $\frac{\mathrm{d}V}{\mathrm{d}t} \alpha A$

So $\frac{\mathrm{d}V}{\mathrm{d}t} = kA, k < 0$ ....(1)

$$V = \frac{4}{3}\pi r^3$$

Differentiate V with respect to r

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$= A\frac{\mathrm{d}r}{\mathrm{d}t} \text{ ....(2)}$$

Equating (1) = (2)

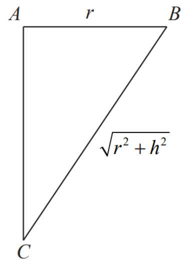
$$kA = A\frac{\mathrm{d}r}{\mathrm{d}t}$$

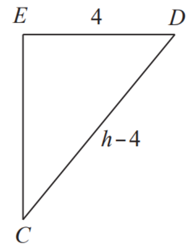
$$\therefore \quad \frac{\mathrm{d}r}{\mathrm{d}t} = k$$

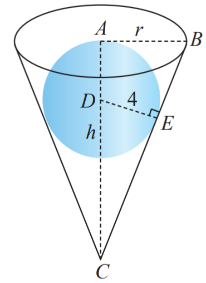
Since $k < 0,$ the radius decreases at a constant rate (Shown)

33
Solution

(a) Using similar triangles,







$$\frac{BC}{BA}=\frac{DC}{DE}$$

$$\frac{\sqrt{r^2+h^2}}{r}=\frac{h-4}{4}$$

$$16(r^2+h^2)=r^2(h-4)^2$$

$$r^2(h^2-8h)=16h^2$$

$$r^2=\frac{15h^2}{h^2-8h}$$

$\therefore$ $\quad r=\dfrac{4h}{\sqrt{h^2-8h}}$ (Shown) or $r=\dfrac{4h}{\sqrt{h^2-8h}}$ (Rejected $\because r>0$)

(b) Let volume of cone be V

$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{4h}{\sqrt{h^2-8h}}\right)^2 h$$

$$=\frac{16\pi}{2}\left(\frac{h^2}{h-8}\right) \text{ .... (1)}$$

Differentiate (1) with respect to r

$$\frac{dV}{dh}=\frac{16\pi}{3}\left(\frac{2h(h-8)-h^2}{(h-8)^2}\right)$$

$$=\frac{16\pi}{3}\left(\frac{h^2-16h}{(h-8)^2}\right)$$

$$=\frac{16\pi(h)(h-16)}{(h-8)^2}$$

For stationary points, $\dfrac{dV}{dh}=0$.

$$\frac{16\pi(h)(h-16)}{(h-8)^2}=0$$

$$16\pi(h)(h-16)=0$$

$\therefore$ $\quad h=0$ (rejected $\because h>0$) or $h=16$

Substitute $h=16$ into (1)

$$V=\frac{16\pi}{3}\left(\frac{16^2}{16-8}\right)$$

$$=\frac{512\pi}{3}$$

h	16^-	16	16^+
Sign of $\dfrac{dV}{dh}$	$-$ve	0	+ve

$\therefore$ V is minimum at $h=16$

$\therefore$ the minimum volume of the cone when $h=16$ is $\dfrac{512\pi}{3}$ cm^3.

(c) Let V_i, A and R be the volume, surface area and radius of the spherical molecule respectively.

$$V_i = \frac{4}{3}\pi R^3 \quad \triangleleft \text{ volume of the spherical molecule}$$

Differentiate V_i with respect to R

$$\frac{dV_i}{dR} = 4\pi R^2$$

$$A = 4\pi R^2 \quad \triangleleft \text{ surface area of the spherical molecule} \text{ .... (2)}$$

Differentiate A with respect to R

$$\frac{dA}{dR} = 8\pi R$$

Given that the surface area of the molecule is decreasing at a constant rate of $\frac{1}{16}\pi \text{ cm}^2/\text{min}$, i.e. $\frac{dA}{dt} = -\frac{\pi}{16}$.

Using Chain Rule,

$$\begin{aligned}\frac{dV_i}{dt} &= \frac{dV_i}{dR} \times \frac{dR}{dA} \times \frac{dA}{dt} \\ &= 4\pi R^2 \times \frac{1}{8\pi R} \times \left(-\frac{1}{16}\pi\right) \\ &= -\frac{\pi R}{32} \text{ .... (3)}\end{aligned}$$

At $t = 0$, substitute $t = 0$ into (2)

$$\begin{aligned}A &= 4\pi(4^2) \\ &= 64\pi\end{aligned}$$

At $t = 4$,

$$\begin{aligned}A &= 64\pi - 4\left(\frac{1}{16}\pi\right) \\ &= \frac{255\pi}{4}\end{aligned}$$

$\therefore$ the surface area of the sphere 4 minutes after it has placed in the jar is $\frac{255\pi}{4}$ cm^2.

Substitute $A = \frac{255\pi}{4}$ into (2)

$$\begin{aligned}4\pi R^2 &= 255\pi \\ \pi R^2 &= \frac{255\pi}{4} \\ R &= \sqrt{\frac{255}{16}}\end{aligned}$$

Substitute $R = \sqrt{\frac{255}{16}}$ into (3)

$$\begin{aligned}\therefore \quad \frac{dV_i}{dt} &= -\frac{\pi}{32}\sqrt{\frac{255}{16}} \\ &= -\frac{\sqrt{255}\pi}{128} \text{ cm}^3/\text{min}\end{aligned}$$

$\therefore$ the exact rate of decrease of volume of the molecule 4 minutes after it had placed in the jar is $\frac{\sqrt{255}\pi}{128}$ cm^3/min.

34
Solution

By Pythagoras' theorem,

$h^2 + x^2 = 3.12^2$

By implicit differentiation w.r.t. t,

$2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$

When $h = 1.2$, $\frac{dx}{dt} = 0.2$ and

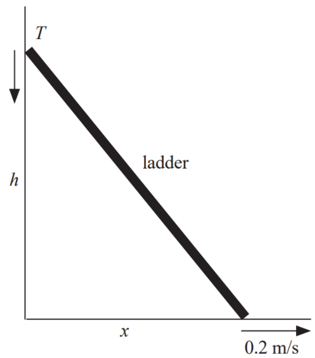
$x^2 = 3.12^2 - 1.2^2$

$x = 2.88 \qquad (\text{Since } x > 0)$

Hence,

$2(1.2)\frac{dh}{dt} + 2(2.88)(0.2) = 0$

$\frac{dh}{dt} = -0.48$



Hence, the top of ladder is sliding down at a rate of 0.48 m/s.

Method 2

By pythagoras' theorem,

$h^2 + x^2 = 3.12^2$

By implicit differentiation w.r.t. x,

$2h\frac{dh}{dx} + 2x = 0 \Rightarrow \frac{dh}{dx} = -\frac{x}{h}$

When $h = 1.2$, $\frac{dx}{dt} = 0.2$ and

$x^2 = 3.12^2 - 1.2^2$

$x = 2.88 \qquad (\text{Since } x > 0)$

Hence, $\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt}$

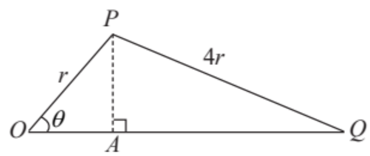
$= -\frac{2.88}{1.2} \times 0.2$

$= -0.48$

Hence, the top of ladder is sliding down at a rate of 0.48 m/s.

35
Solution

(a)



From the diagram,
using trigonometric ratio,
$OA = r\cos\theta$ and $AP = r\sin\theta$

$OQ = OQ - OA$
$\therefore\ OQ = x - r\cos\theta$

By Pythagoras Theorem,

$$AP^2 + AQ^2 = PQ^2$$
$$r^2\sin^2\theta + (x - r\cos\theta)^2 = 16r^2$$
$$x - r\cos\theta = \sqrt{16r^2 - r^2\sin^2\theta} \quad \text{(since } x - r\cos\theta > 0\text{)}$$
$$x = r\left(\cos\theta + \sqrt{16 - \sin^2\theta}\right) \quad \text{(Shown)}$$

Alternative Method
Using Cosine rule,

$$(4r)^2 = r^2 + x^2 - 2rx\cos\theta$$
$$x^2 - 2rx\cos\theta = 15r^2$$
$$(x - r\cos\theta)^2 - r^2\cos^2\theta = 15r^2$$
$$(x - r\cos\theta)^2 = 15r^2 + r^2\cos^2\theta$$
$$(x - r\cos\theta)^2 = 15r^2 + r^2(1 - \sin^2\theta)$$
$$(x - r\cos\theta)^2 = r^2(16 - \sin^2\theta)$$
$$x - r\cos\theta = r\sqrt{16 - \sin^2\theta} \quad \text{(Since } x - r\cos\theta > 0\text{)}$$

$$\therefore x = r\left(\cos\theta + \sqrt{16 - \sin^2\theta}\right) \quad \text{(Shown) .... (1)}$$

(b) Differentiate (1) with respect to θ

$$\frac{dx}{d\theta} = r\left(-\sin\theta + \frac{1}{2}(16 - \sin^2\theta)^{-\frac{1}{2}}(-2\sin\theta\cos\theta)\right)$$

Using Chain Rule,

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$
$$= r\left(-\sin\theta + \frac{1}{2}(16 - \sin^2\theta)^{-\frac{1}{2}}(-2\sin\theta\cos\theta)\right)\frac{d\theta}{dt}$$
$$= -r\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{16 - \sin^2\theta}}\right)\frac{d\theta}{dt}$$

(c) Given that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.5$ rad/s and when $\theta = \frac{2\pi}{3}$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -r\left(\frac{\sqrt{3}}{2}\right)\left(1 + \frac{\left(-\frac{1}{2}\right)}{\sqrt{16 - \left(\frac{3}{4}\right)}}\right)\left(\frac{1}{2}\right)$$

$$\approx -0.378r \text{ cm/s}$$

Q is moving towards O at a rate of $0.378r$ cm/s.

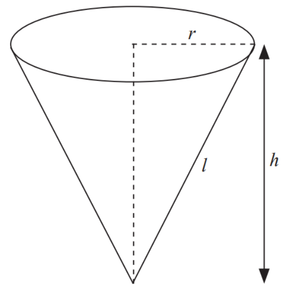
36
Solution

(a) Let l be the slant height of the pot.

By Pythagoras Theorem,

$$l^2 = h^2 + r^2$$

$$l = \sqrt{h^2 + r^2} \text{ .... (1)}$$



Let A be the curved surface area of the pot

$$A = \pi r l \text{ .... (2)}$$

Substitute (1) into (2)

$$A = \pi r\sqrt{h^2 + r^2}$$

$$h = \sqrt{\left(\frac{A}{\pi r}\right)^2 - r^2} \text{ .... (3)}$$

Volume of the pot, $V = \frac{1}{3}\pi r^2 h$ .... (4)

Substitute (3) into (4)

$$V = \frac{1}{3}\pi r^2\sqrt{\left(\frac{A}{\pi r}\right)^2 - r^2}$$

Given that the pot has a fixed external curved surface area of $a\pi$ cm^2, $A = a\pi$

$$V = \frac{1}{3}\pi r^2\sqrt{\left(\frac{a\pi}{\pi r}\right)^2 - r^2}$$

$$V = \frac{1}{3}\pi r^2\sqrt{\left(\frac{a}{r}\right)^2 - r^2}$$

$$V = \frac{1}{3}\pi r^2\sqrt{\frac{a^2 - r^2}{r^2}}$$

$$= \frac{1}{3}\pi r\sqrt{a^2 - r^4}$$

$$9V^2 = \pi^2 r^2 (a^2 - r^4) \quad \text{(shown) .... (5)}$$

Differentiate (5) with respect to r

$$18V\frac{\mathrm{d}V}{\mathrm{d}r} = 2\pi^2 a^2 r - 6\pi^2 r^5$$

At maximum volume of the pot, when $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$.

$$\pi^2 a^2 r - 3\pi^2 r^5 = 0$$

$$r(a^2 - 3r^4) = 0$$

$$r = \sqrt[4]{\frac{a^2}{3}} \quad \text{or} \quad r = 0 \quad \text{(Rejected, since } r > 0\text{)}$$

Use First Derivative Test to show minimum

$$18V\frac{\mathrm{d}V}{\mathrm{d}r} = 2\pi^2a^2r - 6\pi^2r^5$$

$$9V\frac{\mathrm{d}V}{\mathrm{d}r} = \pi^2a^2r - 3\pi^2r^5 \quad \text{.............................} \ (6)$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\pi^2r(a^2-3r^4)}{9V}$$

$$= \frac{-\pi^2r}{3V}\left(r^2+\frac{a}{\sqrt{3}}\right)\left(r+\sqrt{\frac{a}{\sqrt{3}}}\right)\left(r-\sqrt{\frac{a}{\sqrt{3}}}\right)$$

$$= -\left[\frac{\pi^2r}{3V}\left(r^2+\frac{a}{\sqrt{3}}\right)\left(r+\sqrt{\frac{a}{\sqrt{3}}}\right)\right]\left(r-\sqrt{\frac{a}{\sqrt{3}}}\right)$$

Since $V > 0, r > 0,$

r	$\left(\sqrt{\frac{a}{\sqrt{3}}}\right)^-$	$\sqrt{\frac{a}{\sqrt{3}}}$	$\left(\sqrt{\frac{a}{\sqrt{3}}}\right)^+$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	$-[+](-) = +ve$	0	$-[+](+) = -ve$
Tangent	/	—	\

$\therefore$ V is maximum at $r = \sqrt[4]{\frac{a^2}{3}}$.

From (5): $9V^2 = \pi^2r^2(a^2-r^4)$

$$V = \frac{1}{3}\pi r\sqrt{a^2-r^4}$$

Substitute $r = \sqrt[4]{\frac{a^2}{3}}$ into $V = \frac{1}{3}\pi r\sqrt{a^2-r^4}$

$$V = \frac{1}{3}\pi r\sqrt{a^2-r^4}$$

$$= \frac{1}{3}\pi\left(\sqrt[4]{\frac{a^2}{3}}\right)\sqrt{a^2-\left(\sqrt[4]{\frac{a^2}{3}}\right)^4} = \frac{1}{3}\pi\left(\sqrt[4]{\frac{a^2}{3}}\right)\sqrt{\frac{2a^2}{3}}$$

$$= \frac{\sqrt{2}\pi}{3}\left(\frac{a^2}{3}\right)^{\frac{3}{4}}$$

$$= \pi\sqrt{2}a^{\frac{3}{2}}3^{-\frac{7}{4}}$$

$\therefore$ the maximum volume is $\pi\sqrt{2}a^{\frac{3}{2}}3^{-\frac{7}{4}}$ cm^3

Alternative Method (Use Second Derivative Test to show minimum)

From (6) $9V\dfrac{dV}{dr} = \pi^2a^2r - 3\pi^2r^5$

$$9\left(\frac{dV}{dr}\right)^2 + 9V\frac{d^2V}{dr^2} = \pi^2a^2 - 15\pi^2r^4 \text{ ....(7)}$$

Substitute $r = \sqrt[4]{\dfrac{a^2}{3}}$ and $\dfrac{dV}{dr} = 0$ into (7):

$$9(0)^2 + 9V\frac{d^2V}{dr^2} = \pi^2a^2 - 15\pi^2\left(\frac{a^2}{3}\right)$$

$$\frac{d^2V}{dr^2} = -\frac{4\pi^2a^2}{9V}$$

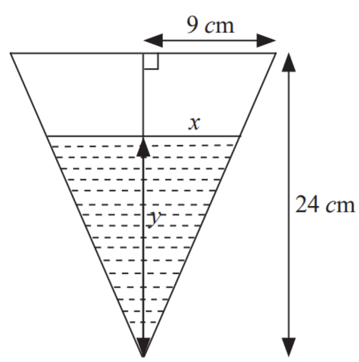
Since $V > 0$, therefore $\dfrac{d^2V}{dr^2} < 0$

$\therefore$ V is maximum when $r = \sqrt[4]{\dfrac{a^2}{3}}$.

(b) Using similar triangles,

$$\frac{x}{9} = \frac{y}{24}$$

$$x = \frac{3y}{8} \text{ .... (1)}$$



Let W be the volume of the water in the pot at time t

$$W = \frac{1}{3}\pi r^2 y \text{ .... (2)}$$

Substitute (1) into (2)

$$= \frac{1}{3}\pi\left(\frac{3}{8}y\right)^2 y$$

$$= \frac{3}{64}\pi y^3 \text{ .... (3)}$$

Differentiate (3) with respect to 3

$$\frac{dW}{dy} = \pi\left(\frac{9}{64}\right)y^2$$

When $y = 1$, substitute $y = 1$ into $\dfrac{dW}{dy} = \pi\left(\dfrac{9}{64}\right)y^2$

$$\frac{dW}{dy} = \frac{9}{64}\pi$$

Using chain rule,

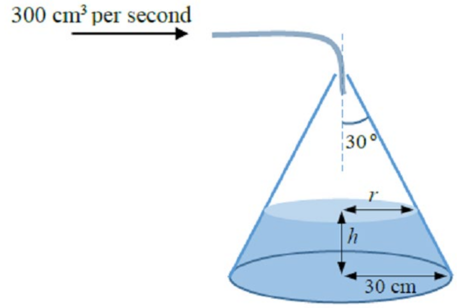
$$\frac{dy}{dt} = \frac{dy}{dW} \times \frac{dW}{dt}$$

$$= \frac{64}{9\pi} \times \left(-\frac{1}{2}\right)$$

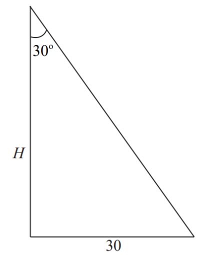
$$= -\frac{32}{9\pi} \text{ cms}^{-1}$$

$\therefore$ the rate of decrease in the depth of water is $\dfrac{32}{9\pi}$ cm per second.

37
Solution

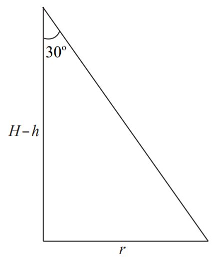


(a) Let H be the height of conical container



$$\tan 30^\circ = \frac{30}{H}$$

$$H = \frac{30}{\tan 30^\circ} = 30\sqrt{3}$$



$$\tan 30^\circ = \frac{r}{30\sqrt{3}-h}$$

$$r = \left(30\sqrt{3}-h\right)\tan 30^\circ = \frac{30\sqrt{3}-h}{\sqrt{3}}$$

V = Volume of cone − Volume of not filled with water

$$V = \frac{1}{3}\pi(30^2)(30\sqrt{3}) - \frac{1}{3}\pi\left(\frac{30\sqrt{3}-h}{\sqrt{3}}\right)^2(30\sqrt{3}-h)$$

$$= 9000\sqrt{3}\pi - \frac{\pi}{9}(30\sqrt{3}-h)^3 \quad \text{(Shown)}$$

(b) Differentiate V with respect to t

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{3\pi}{9}\left(30\sqrt{3}-h\right)^2\left(-\frac{\mathrm{d}h}{\mathrm{d}t}\right)$$

$$= \frac{\pi}{3}\left(30\sqrt{3}-h\right)^2\frac{\mathrm{d}h}{\mathrm{d}t}$$

Water is poured at a constant rate of 300 cm^3 per second, i.e. $\frac{\mathrm{d}V}{\mathrm{d}t} = 300$

$$300 = \frac{\pi}{3}\left(30\sqrt{3}-5\sqrt{3}\right)^2\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$300 = \frac{\pi}{3}\left(25\sqrt{3}\right)^2\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{300}{625\pi} = \frac{12}{25\pi}$$

Hence the rate of change of depth of water is $\frac{12}{25\pi}$ cm s^{-1}.

Alternate Method

Differentiate V with respect to h

$$\frac{\mathrm{d}V}{\mathrm{d}h} = -\frac{\pi}{3}\left(30\sqrt{3} - h\right)^2(-1)$$

$$= \frac{\pi}{3}\left(30\sqrt{3} - h\right)^2$$

Using Chain Rule,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \frac{1}{\frac{\pi}{3}\left(30\sqrt{3} - h\right)^2} \times (300)$$

$$= \frac{900}{\pi\left(30\sqrt{3} - h\right)^2}$$

When $h = 5\sqrt{3}$,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{900}{\pi\left(30\sqrt{3} - 5\sqrt{3}\right)^2}$$

$$= \frac{12}{25\pi} \text{ cm s}^{-1}$$

Hence rate of change of depth of water is $\frac{12}{25\pi}$ cm s^{-1}.

38
Solution

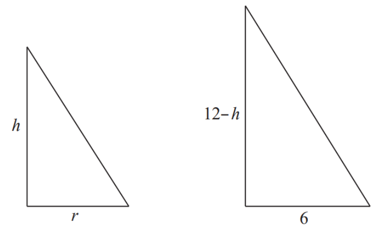
(a)(i)

Using similar triangles,

$$\frac{12-h}{12}=\frac{r}{6}$$

$$12-h=2r$$

$$h=12-2r \text{ .... (1)}$$



(a)(ii)

Volume of the smaller cone $V=\frac{1}{3}\pi r^2 h$ .... (2)

Substitute (1) into (2)

$$V=\frac{1}{3}\pi r^2(12-2r)$$

$$=4\pi r^2-\frac{2}{3}\pi r^3$$

Differentiate V with respect to r

$$\frac{\mathrm{d}V}{\mathrm{d}r}=8\pi r-2\pi r^2 \text{ .... (3)}$$

At maximum, $\frac{\mathrm{d}V}{\mathrm{d}r}=0$

$$8\pi r-2\pi r^2=0$$

$$2\pi r(4-r)=0$$

$$r=4 \quad \text{or} \quad r=0 \text{ (Rejected, since } r>0\text{)}$$

Substitute $r=4$ into (1)

$\therefore \quad h=12-2(4)$

$h=4$

Differentiate (3) with respect to r

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=8\pi-4\pi r$$

When $r=4$

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=8\pi-4\pi(4)$$

$$=-8\pi<0$$

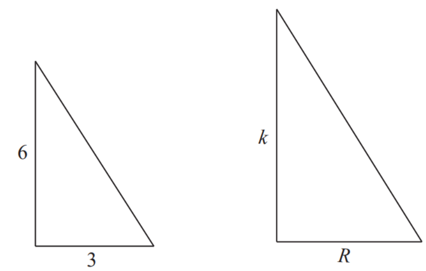
$\therefore$ $r=4$ and $h=4$ is maximum such that the smaller cone has the largest possible volume.

(b)(i) Let the radius of the surface of the water in the smaller cone at time t seconds be R.

Using similar triangles

$$\frac{3}{R} = \frac{6}{k}$$

$$\frac{3}{6}k = R \text{ .... (1)}$$



Volume of the smaller cone, $V = \frac{1}{3}\pi r^2 h$ .... (2)

Substitute (1) into (2)

$$V = \frac{1}{3}\pi\left(\frac{3}{6}k\right)^2 k$$

$$= \frac{1}{3}\pi\left(\frac{1}{4}k^2\right)k$$

$\therefore \quad V = \frac{1}{12}\pi k^3$ .... (3) (Shown)

Differentiate (3) with respect to k

$$\frac{\mathrm{d}V}{\mathrm{d}k} = \frac{\pi k^2}{4}$$

Using Chain Rule,

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{\mathrm{d}k}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \frac{4}{\pi k^2} \times \frac{\pi}{16}$$

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{4k^2} \text{ .... (4)}$$

Integrate both sides with respect to t

$$\int k^2 \,\mathrm{d}k = \int \frac{1}{4}\,\mathrm{d}t$$

$\frac{k^3}{3} = \frac{1}{4}t + c$, where c is a constant.

When $t = 0$ and $k = 0$

$\therefore \; c = 0.$

Hence, $\dfrac{k^3}{3} = \dfrac{1}{4}t$

When $t = \dfrac{32}{3}$

$$\frac{k^3}{3} = \frac{1}{4}\left(\frac{32}{3}\right)$$

$$\frac{k^3}{3} = \frac{8}{3}$$

$$k^3 = 8$$

$$k = 2$$

Substitute $k = 2$ into (4)

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{4(2)^2}$$

$$= \frac{1}{16}$$

$\therefore$ the rate of k increasing at $\dfrac{32}{3}$ seconds is $\dfrac{1}{16}$ cm per second.

39
Solution

(a) Let x = distance of the car from the junction P and y = distance between the car and truck

By Cosine Rule,

$y^2 = x^2 + 48^2 - 2x(48)\cos 120^\circ$

$y^2 = x^2 + 48x + 2304$ (shown) .... (1)

(b) Differentiate (1) with respect to t

$$2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 48\frac{\mathrm{d}x}{\mathrm{d}t}$$

When $x = 15$, substitute $x = 15$ into (1)

$y^2 = x^2 + 2304 + 48x$

$y^2 = 15^2 + 2304 + 48(15) = 3249$

$y = 57$

The rate of change of the car is $\frac{\mathrm{d}x}{\mathrm{d}t} = -60$ because it is traveling toward P.

$$2(57)\frac{\mathrm{d}y}{\mathrm{d}t} = (2(15) + 48)\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -41\frac{1}{19}$$

$$= -41.1.\,(3\text{sf})$$

Hence, the distance between the car and the truck is decreasing at a rate of $41.1\,\mathrm{km/h}$.